

STATE SPACES OF JORDAN ALGEBRAS

Erik M. Alfsen and Frederic W. Shultz

Introduction

The purpose of this paper is to give a geometric characterization of the state spaces of the class of normed Jordan algebras named JB-algebras in [3]. Recall in this connection that by the generalized Gelfand-Neumark theorem of [3] the study of JB-algebras can be reduced to the study of Jordan algebras of self-adjoint operators on a Hilbert space and the exceptional algebra M_3^8 . One of the most important examples of a JB-algebra is the self-adjoint part of a C^* -algebra; thus the properties we establish for state spaces of JB-algebras also give information about state spaces of C^* -algebras.

It is known that state spaces of JB-algebras are strongly spectral compact convex sets in the terminology of [2], but this property alone does not characterize state spaces of JB-algebras. (For example, any two-dimensional strictly convex and smooth compact convex set is spectral, but it is a state space only if its boundary is an ellipse.) In the present paper a characterization is obtained by adding two geometric conditions: symmetry and the Hilbert ball property.

First, a spectral convex set K is said to be "symmetric" if it is symmetric with respect to each set $\text{co}(F \cup F^\#)$ where F is a

The final result is then established in § 7, where the results from § 6 are globalized by the existence of a separating family of "type - I representations".

Acknowledgements. Part of the present paper was completed while the authors were visiting the University of California, Berkeley. Both authors would like to express their gratitude to the Mathematics Department in Berkeley and especially to professors William Bade and Marc Rieffel for their hospitality during the time each author spent in Berkeley.

The first author would like to express his gratitude to the Norwegian Council for Science and the Humanities (NAVF) for a travel grant which made the visit to Berkeley possible. The second author gratefully acknowledges support from a research grant from Wellesley College during a year spent on leave.

§ 1. Preliminaries on non-commutative spectral theory.

In this section we will summarize the main results from [2] which are needed in the sequel, and we will also establish some useful new results on spectral theory. Our setting in this section will be that of [2], i.e. we shall consider an order-unit space (A, e) and a base-norm space (V, K) in separating order and norm duality.

Following [2] we say that two positive projections P, Q on A (or on V) are quasicomplementary (q.c.) if

$$(1.1) \quad \ker^+ P = \operatorname{im}^+ Q, \quad \operatorname{im}^+ P = \ker^+ Q.$$

A weakly (i.e. $\sigma(A, V)$ -) continuous positive projection $P: A \rightarrow A$ with $\|P\| \leq 1$ is said to be a P-projection if there exists a (necessarily unique) weakly continuous positive projection $P': A \rightarrow A$ with $\|P'\| \leq 1$ such that P, P' are q.c. and the dual projections P^*, P'^* (on V) are also q.c. (See Theorems 1.8 and 2.5 of [2] for alternative characterizations of P-projections).

To every P-projection P is associated a projective unit $u_P = Pe$ and a projective face

$$F_P = K \cap \operatorname{im} P^* = \{p \in K \mid \langle Pe, p \rangle = 1\}.$$

(See Theorem 3.5 of [2] for a geometric characterization of projective faces). The sets of P-projections on A , projective units in A , and projective faces of K are denoted by \mathcal{P}, \mathcal{U} , and \mathcal{F} , respectively. In \mathcal{P} an ordering is defined by $P \leq Q$ when $PQ = QP = P$; this is equivalent to the natural ordering of the corresponding elements of \mathcal{U} and \mathcal{F} , i.e. to $Pe \leq Qe$ and $F_P \subseteq F_Q$ (cf. [2; Lem. 2.16]). Thus the sets $\mathcal{U}, \mathcal{P}, \mathcal{F}$ are order

isomorphic under the maps $Pe \leftrightarrow P \leftrightarrow F_P$ [2;Th.2.17]. Using these maps one can transfer the notion of a quasicomplement from \mathcal{P} to \mathcal{U} and \mathcal{F} . Specifically, let $u \in \mathcal{U}$ and $F \in \mathcal{F}$, say $u = Pe$ and $F = F_P$ with $P \in \mathcal{P}$; then $u' = P'e = e - u$ and $F^\# = F_{P'} = K \cap \text{im } P'^*$. (The symbol $F^\#$ is chosen to avoid confusion with the ordinary complement F' of convexity theory. Generally $F^\# \subseteq F'$, cf. [2;p.15]).

A P -projection P is said to be compatible with an element $a \in A$ if $a = (P+P')a$. Note that P is compatible with a iff P' is. It can be proved [2;Prop.5.2] that a P -projection P is compatible with a projective unit $u = Qe$ where $Q \in \mathcal{P}$ iff P and Q commute; in this case we say that P and Q are compatible. Note that if P, Q are compatible, then PQ is also a P -projection, in fact it is the g.l.b. of P and Q in the ordering of \mathcal{P} [2;Prop.5.2]. It is also easily verified that if $P \preceq Q$, then P and Q are compatible. A P -projection P is said to be bicompatible with an element $a \in A$ if P is compatible with a and with all $Q \in \mathcal{P}$ compatible with a . A P -projection is said to be central if it is compatible with all $a \in A$. These notions are all transferred from \mathcal{P} to \mathcal{U} and \mathcal{F} by the natural isomorphisms.

To achieve a spectral theory one must impose an axiom to ensure that there are "sufficiently many" projective faces of K . Specifically, (A, e) and (V, K) are said to be in spectral duality if A is pointwise monotone σ -complete (viewed as a function space on K) and if there exists for every $a \in A$ and every $\lambda \in \mathbb{R}$ a projective face F such that

$$(1.2) \quad a \leq \lambda \text{ on } F, \quad a > \lambda \text{ on } F^\#,$$

$$(1.3) \quad F \text{ is bicompatible with } a.$$

It can be proved that the requirements (1.2) and (1.3) determine $F \in \mathcal{F}$ uniquely [2; Lem.7.1]. The corresponding projective unit is termed the spectral unit of a for the value λ , and it is denoted by e_λ^a or simply by e_λ when there is no need to specify the element $a \in A$.

If (A, e) and (V, K) are in spectral duality, then \mathcal{P} (hence also \mathcal{U} and \mathcal{F}) is a σ -complete orthomodular lattice. (See Theorem 4.5 of [2]).

If (A, e) and (V, K) are in spectral duality, then every $a \in A$ can be written as an abstract Stieltjes integral $a = \int \lambda de_\lambda$ defined by the corresponding (numerical) integrals

$$(1.4) \quad \langle a, \rho \rangle = \int \lambda d\langle e_\lambda, \rho \rangle \quad \text{all } \rho \in K.$$

(Cf. [2; Th.6.8]). Accordingly we will term the family $\{e_\lambda\}$ the spectral resolution of the given element $a \in A$.

Now one can define a functional calculus in A by writing $\varphi(a) = \int \varphi(\lambda) de_\lambda$ for every φ in the class \mathcal{B} of bounded Borel functions on \mathbb{R} . This functional calculus will satisfy all customary requirements ((8.22)-(8.27) in [2]), and it will be the only such calculus which is "extreme point preserving" in that $\chi_E(a)$ is an extreme point of $A_1^+ = \{a \in A \mid 0 \leq a \leq e\}$ for every Borel set E (i.e. for every extreme point χ_E of \mathcal{B}_1^+) [2; Th.8.9]. Note in this connection that the extreme points of A_1^+ are exactly the projective units, and they are in turn just those elements $a \in A$ which are "idempotent under squaring", i.e. $a^2 = a$ where $a^2 = \varphi(a)$ with $\varphi(\lambda) = \lambda^2$ for $\lambda \in \mathbb{R}$ [2; Prop.8.7]. (Observe that spectral duality is essential for the identification of projective units with extreme points. In more general cases the projective units can form a proper subset of the extreme points of A_1^+ , cf. [2; p.14]).

Assuming that (A, e) and (V, K) are in spectral duality, we can extend the notion of compatibility to arbitrary pairs of elements of A . For given $a, b \in A$ we say that a and b are compatible if all pairs e_λ^a, e_μ^b from their spectral resolutions are compatible. Then we define an abelian subspace of A to be a norm closed subspace containing e , closed under the map $a \mapsto a^2$, and all of whose members are mutually compatible. Every abelian subspace of A is isometrically isomorphic to $C(X)$ for X compact Hausdorff when equipped with the product

$$(1.5) \quad a \cdot b = \frac{1}{2}((a+b)^2 - a^2 - b^2) .$$

(Cf. [2; Prop. 9.8]). For every subset of A consisting of mutually compatible elements (in particular for every single element of A) there is a smallest abelian subspace of A containing it (and a smallest weakly closed abelian subspace containing it). The set $Z(A)$ of elements in A compatible with every element of A form a weakly closed abelian subspace called the center of A . (Note that this definition of center is consistent with that of [23], cf. [2; Th. 9.19], and note also that a P -projection is central in the previous sense of the word iff $Pe \in Z(A)$, cf. [2; p. 79]).

In most of the important applications the spaces A and V will satisfy the requirement $A \cong V^*$. Then the lattice \mathcal{U} (and hence \mathcal{P} and \mathcal{F}) is complete [2; Cor. 12.5]. Recall also that when (A, e) and (V, K) are in spectral duality with $A \cong V^*$, then every A -semi-exposed face of K is A -exposed, therefore projective [2; Cor. 12.4]. Hence in this case the collection of projective faces of K coincides with the collection of A -semi-exposed faces of K , and so it is closed under arbitrary inter-

sections. It follows that every subset $E \subseteq K$ is contained in a smallest projective face $F = F(E)$. If $E = \{\rho\}$ for some $\rho \in K$, we write simply $F(\rho)$. For $\rho \in V^+ \setminus \{0\}$ we extend this notation by writing $F(\rho) = F(\frac{\rho}{\|\rho\|})$, and we set $F(0) = \emptyset$.

In the applications to convexity theory one starts out with a convex set K which can be embedded in a linear space V (uniquely up to a linear isomorphism) in such a way that (V, K) becomes a base-norm space. (Recall from [1 ; p.77] that (V, K) is a base-norm space if the affine span of K is a hyperplane not passing through the origin and $\text{co}(KU - K)$ is radially compact, this set being the unit ball of the base-norm). As usual we will denote the space of bounded affine functions on K by $A^b(K)$.

We will say that a convex set K is spectral if it can be embedded in a linear space V in such a way that (V, K) becomes a base-norm space in spectral duality with (A, e) where $A = A^b(K) \cong V^*$. When working with spectral convex sets, we will always assume that this embedding is performed, and we will make free use of notions from general spectral theory (e.g. that of a "projective face") which will then refer to the duality of (A, e) and (V, K) .

The above definition is more general than that of [2] which is confined to compact convex sets. Note, however, that the new definition will agree with the old one when K is compact, since every compact convex set can be embedded in a locally convex space V in such a way that (V, K) becomes a base-norm space (the "regular embedding" [1 ; Ch.II, § 2]).

Following [2] we say that a compact convex set K is strongly spectral if it is spectral and if in addition e_λ^a is u.s.c. in the given topology of K for all $\lambda \in \mathbb{R}$ and all a in the space $A(K)$ of continuous affine functions on K . A spectral

compact convex set K is strongly spectral iff $A(K)$ is closed under functional calculus by continuous functions [2; Th.10.6]. In particular, if K is strongly spectral, then $A(K)$ is closed under the squaring map $a \mapsto a^2$. Examples of spectral compact convex sets are the closed unit balls of L_p for $1 < p < \infty$ in the weak topology and also all Choquet simplexes. The former are all strongly spectral, the latter are strongly spectral iff they have closed extreme boundaries (i.e. if they are Bauer simplexes). [2; Ths.10.4-10.5 and Prop.10.9].

Now let (A, e) and (V, K) be in spectral duality and suppose that $A \cong V^*$. For a given P -projection P with associated projective face $F \subset K$ we consider the restriction ψ_F of $(P+P')^*$ to K . This map can be shown to be the unique affine retraction of K onto $\text{co}(F \cup F^\#)$ [2; Th.3.8]. If F is a split face of K (this occurs exactly when P is a central P -projection [2; Prop.10.2]), then K is the union of all line segments $[\sigma, \tau]$ with $\rho \in F$ and $\sigma \in F^\#$. For a general projective face F , then K is the union of the fibers $\psi_F^{-1}([\rho, \sigma])$ with $\rho \in F$ and $\sigma \in F^\#$. Note that two different fibers will either be disjoint or they will meet at an "end point" $\rho \in F$ or $\sigma \in F^\#$. Later on we shall see that the geometry of these fibers holds important information about K .

An important special case is obtained by taking A to be the self-adjoint part of a von Neumann algebra \mathcal{O} with identity element e , and V to be the self-adjoint part of the predual space \mathcal{O}_* with K the normal state space. Then (A, e) and (V, K) are in spectral duality with $A = V^*$, and the P -projections are precisely the maps $a \mapsto pap$ with p a self-adjoint projection in \mathcal{O} [2; Prop.11.4]. In this case the notions of

compatibility and bicompatibility will coincide with commutation and bicommutation [2; Cor.11.3]. Applying these results to the enveloping von Neumann algebra, one easily sees that the state space of a C^* -algebra is a strongly spectral compact convex set [2; Th.11.6].

After this summary of results from [2] we shall present some new results on non-commutative spectral theory which will be needed in the sequel. Our first proposition will be a new characterization of spectral duality. In this connection we recall that if (A, e) and (V, K) are in separating order and norm duality such that A is pointwise monotone σ -complete and every A -exposed face of K is projective (this is implied by spectral duality [2; Prop.6.2]), then every $a \in A^+$ admits a smallest element $r(a) \in \mathcal{U}$ such that $a \in \text{face}(r(a))$. (Here $\text{face}(r(a))$ denotes the face of A^+ generated by $r(a)$, cf. [2; Prop.4.7]). In fact, $r(a)$ is the unique element of \mathcal{U} such that for $\rho \in K$:

$$(1.6) \quad \langle r(a), \rho \rangle = 0 \iff \langle a, \rho \rangle = 0.$$

Recall also that two elements $a, b \in A^+$ are said to be orthogonal, in symbols $a \perp b$, if $r(a) + r(b) \leq e$, or equivalently if the P -projections corresponding to $r(a)$ and $r(b)$ annihilate each other [2; Prop.4.4].

Proposition 1.1. Let (A, e) and (V, K) be in separating order and norm duality such that A is pointwise monotone σ -complete and every A -exposed face of K is projective. Then the two spaces will be in spectral duality iff every $a \in A$ admits a unique decomposition $a = a^+ - a^-$ where $a^+, a^- \in A^+$ and $a^+ \perp a^-$.

Proof. 1.) To prove that the condition is sufficient, we assume that every $a \in A$ has a decomposition of the type described. By Proposition 6.3 of [2] (A, e) and (V, K) are in "weak" spectral duality, in that they satisfy all requirements for spectral duality except the bicompatibility in (1.3) which is replaced by compatibility only. But by Theorem 7.5 of [2] the desired bicompatibility will follow if we can prove that for any given $a \in A$ and $\lambda \in \mathbb{R}$ there is a unique $F \in \mathcal{F}$ compatible with a which satisfies (1.2). Since we can replace a by $a - \lambda e$, we may as well assume $\lambda = 0$. Thus, we shall prove that there exists only one $F \in \mathcal{F}$ compatible with a such that

$$(1.7) \quad a \leq 0 \text{ on } F, \quad a > 0 \text{ on } F^\#.$$

Now let $F \in \mathcal{F}$ and $P \in \mathcal{P}$ correspond to the projective unit $e - r(a^+)$. Then $a^+ = P'a$ and $a^- = -Pa$, and by (1.6) we have

$$(1.8) \quad F = \{\rho \in K \mid \langle r(a^+), \rho \rangle = 0\} = \{\rho \in K \mid \langle a^+, \rho \rangle = 0\}.$$

Hence $a^+ > 0$ on $F^\# \subset K \setminus F$. Thus since $a = P'a = a^+$ on $F^\#$ and $a = Pa = -a^-$ on F , then the projective face F must satisfy the relations of (1.7).

Now suppose that G is any projective face of K compatible with a and satisfying (1.7) with G in place of F . (Such a G exists by weak spectral duality). Let $Q \in \mathcal{P}$ correspond to G . Then $a = Q'a - (-Qa)$ will be a decomposition of the type mentioned in the theorem, so $a^+ = Q'a$. It follows that $a^+ = 0$ on G , and so by (1.8) $G \subseteq F$. Now F and G are compatible, and it follows by the argument of [2; Lem.7.1] that F is the only member of \mathcal{F} which is compatible with a and satisfies (1.7).

2.) To prove necessity we assume (A, e) and (V, K) in spectral duality. By Proposition 6.3 of [2] every $a \in A$ admits a decomposition $a = a^+ - a^-$ with $a^+, a^- \in A^+$ and $a^+ \perp a^-$. Taking F and P to be the projective face and P -projection corresponding to $e - r(a^+)$ and arguing as before, we conclude that F is a projective face compatible with a for which (1.7) holds. Spectral duality implies that such projective faces are uniquely determined [2; Th.7.2], and so $F = F_0$ and $P = P_0$ where $F_0 \in \mathcal{F}$ and $P_0 \in \mathcal{P}$ correspond to the spectral unit e_0^a . Now $a^+ = P'a = P_0'a$. Hence a^+ is uniquely determined, and we are done. \square

Remark. The above Proposition 1.1 may be considered a supplement to Proposition 6.3 of [2], stating that uniqueness of the decomposition $a = a^+ - a^-$ is exactly what is needed to pass from weak spectral duality to spectral duality.

Our next result is a technical lemma. For the proof of this lemma we recall two useful formulas from [2]. If u_1, u_2, \dots, u_n are mutually orthogonal projective units, then by [2; (4.13)]:

$$(1.9) \quad \bigvee_{i=1}^n u_i = u_1 + \dots + u_n.$$

Secondly, by the orthomodular identity [2; Th.4.5 (iv)] and (1.9), the following holds for projective units u, v :

$$(1.10) \quad u \leq v \text{ implies } v - u = v \wedge u'.$$

Lemma 1.2. If (A, e) and (V, K) are in spectral duality, then for given $a \in A^+$ and $P \in \mathcal{P}$

$$(1.11) \quad r(Pa) = (r(a) \vee P'e) \wedge Pe.$$

Proof. We claim that it suffices to prove

$$(1.12) \quad r(a) \vee P'e = r(Pa) + P'e .$$

In fact, given this inequality we write $u = P'e$ and $v = r(a) \vee P'e$. Then also $v = r(Pa) + P'e$, and since $u \leq v$, we obtain from (1.10)

$$\begin{aligned} (r(a) \vee P'e) \wedge Pe &= v \wedge u' = v - u \\ &= (r(Pa) + P'e) - P'e = r(Pa) . \end{aligned}$$

To prove (1.12) we denote by Q and R the P -projections corresponding to $r(a)$ and $r(Pa)$, respectively. Note that since P' and $P' \vee Q$ are compatible, then P and $P' \vee Q$ will also be. Hence P and $P' \vee Q$ commute, and so

$$(P' \vee Q)Pa = P(P' \vee Q)a = Pa ,$$

since $a \in \text{face}(r(a)) = \text{im}^+ Q \subseteq \text{im}(P' \vee Q)$. (Recall that for $P_0 \in \mathcal{P}$ $\text{im}^+ P_0$ equals the face of A^+ generated by $P_0 e$ [2; Cor.2.12]). Now $Pa \in \text{im}(P' \vee Q)$, and so

$$r(Pa) \leq (P' \vee Q)e = (P'e) \vee r(a) .$$

Clearly $r(Pa) \perp P'e$ (since $r(Pa) \leq Pe$); therefore (by (1.9)):

$$(1.13) \quad r(Pa) + P'e = r(Pa) \vee P'e \leq (P'e) \vee r(a) .$$

To prove the other half of (1.12) we note that $r(Pa) \leq Pe$ implies $R \leq P$. Hence $P \wedge R' = R'P$ and $RP = P$. Thus we have

$$(P \wedge R')a = R'Pa = R'(RPa) = 0 .$$

This implies (since $\ker^+(P \wedge R') = \text{im}^+(P \wedge R')'$):

$$(P' \vee R)a = (P \wedge R')'a = a .$$

Hence $a \in \text{im}(P' \vee R) = \text{face}((P' \vee R)e)$, and so

$$r(a) \leq (P' \vee R)e = (P'e) \vee r(Pa) .$$

Hence we have (cf. (1.13)):

$$(1.14) \quad (P'e) \vee r(a) \leq (P'e) \vee r(Pa) = P'e + r(Pa) .$$

By (1.13) and (1.14) the proof is complete. \square

One can dualize the proof of Lemma 1.2 by replacing P by P^* , a by σ (where $\sigma \in V^+ \setminus \{0\}$), $r(a)$ by $F(\sigma)$, and so on. (Recall that $F(\sigma)$ is the projective face of K generated by $\frac{\sigma}{\|\sigma\|}$). This gives:

Corollary 1.3. Let (A, e) and (V, K) be in spectral duality with $A = V^*$. For given $P \in \mathcal{P}$ with corresponding projective face $F = K \cap \text{im } P^*$ and for each $\sigma \in V^+$ one has

$$(1.15) \quad F(P^* \sigma) = (F(\sigma) \vee F^\#) \cap F .$$

We now turn to the problem of relativizing spectral duality.

Proposition 1.4. Let (A, e) and (V, K) be in spectral duality with $A = V^*$. Let $P_0 \in \mathcal{P}$, $A_0 = \text{im } P_0$, $e_0 = P_0 e$, $V_0 = \text{im } P_0^*$, and $K_0 = V_0 \cap K$. Then (A_0, e_0) and (V_0, K_0) are in spectral duality for the induced pairing and $A_0 = V_0^*$. The P -projections on A_0 for this duality are the maps $P|_{A_0}$ with $P \in \mathcal{P}$, $P \leq P_0$ (such maps leave A_0 invariant); for such a P -projection the quasicomplement is $P'|_{A_0}$ (which is the same as $P'|_{A_0} = P' \wedge P_0|_{A_0}$). The projective faces of K_0 are just those $F \in \mathcal{F}$ such that $F \subseteq K_0$ and the projective units in A_0 are just those $u \in \mathcal{U}$ such that $u \leq e_0$.

Proof. By [2; Prop.2.11] (A_0, e_0) is an order-unit space, and by [2; Prop. 2.14] (V_0, K_0) is a base-norm space. We omit the easy verification that they are in separating order and norm duality with $A_0 = V_0^*$. Note, the latter fact implies that A_0 is pointwise monotone complete.

Recall that if $P \in \mathcal{P}$ and $P \preceq P_0$, then P and P_0 are compatible, and so are P' and P_0 . Moreover, $P = P_0 P$ and $P' \wedge P_0 = P' P_0 = P_0 P'$. It follows that P and P' leave A_0 invariant, and clearly $P|_{A_0}$ and $P'|_{A_0}$ are quasicomplementary. Also $P|_{A_0}$ and $P'|_{A_0}$ are seen to be positive, of norm at most 1, weakly continuous in the induced duality of A_0 and V_0 , and with dual projections $P^*|_{V_0}$ and $P'^*|_{V_0}$. The latter two projections on V_0 are seen to be quasicomplementary. Hence for each $P \in \mathcal{P}$ such that $P \preceq P_0$, the restriction $P|_{A_0}$ will be a P -projection with quasicomplement $P'|_{A_0}$ on A_0 .

In order to apply Proposition 1.1 we will now verify that each A_0 -exposed face of K_0 is projective (in the duality of (A_0, e_0) and (V_0, K_0)). For a given A_0 -exposed face F of K_0 there exists $a \in A_0^+$ such that $a = 0$ on F , $a > 0$ on $K_0 \setminus F$. We define $b = a + P_0' e = a + e - e_0 \in A^+$. For $\rho \in K$ we then have $\langle b, \rho \rangle = 0$ iff $\langle a, \rho \rangle = \langle P_0' e, \rho \rangle = 0$ which in turn is equivalent to $\langle a, \rho \rangle = 0$ and $\rho \in K_0$, i.e. to $\rho \in F$. Thus, F will be an A -exposed face of K and therefore also a projective face of K [2; Prop.6.2]. Let $P \in \mathcal{P}$ correspond to F , i.e. let $F = K \cap \text{im}^* P$. Clearly $P \preceq P_0$ since $F \subseteq K_0$. Therefore $P|_{A_0}$ is a P -projection on A_0 and it follows from the equalities

$$F = K \cap \text{im} P^* = K \cap (\text{im} P_0^*) \cap (\text{im} P^*) = K \cap \text{im}(P^*|_{V_0})$$

that F is a projective face of K_0 corresponding to the P -pro-

jection $P|_{A_0}$ (with dual $P^*|_{V_0}$).

From the above argument it also follows that the projective faces of K_0 are precisely those projective faces of K which are contained in K_0 , that the P -projections on A_0 are precisely the maps $P|_{A_0}$ with $P \in \mathcal{P}$ and $P \leq P_0$, and that the projective units in A_0 are precisely those projective units in A which are majorized by e_0 . Hence it only remains to verify the criterion for spectral duality given in Proposition 1.1.

To this end we consider an arbitrary $a \in A_0$, having the unique decomposition $a = a^+ - a^-$ with $a^+, a^- \geq 0$ and $a^+ \perp a^-$ in the duality of (A, e) and (V, K) . We will show that a^+ and a^- are actually in A_0 . This will complete the proof, for by the above results the definition of $r(a^+)$ and $r(a^-)$, and hence of the relation $a^+ \perp a^-$, will be the same in the duality of (A_0, e_0) and (V_0, K_0) as in the duality of (A, e) and (V, K) .

Since (A_0, e_0) is an order-unit space for the relativized norm, then $a \leq \|a\|e_0$. Clearly a and e_0 are compatible (since $a \in A_0 = \text{im } P_0$ and P_0 are compatible). By [2 ; Prop.9.3] a^+ is the l.u.b. of a and 0 among those elements compatible with a , so we conclude that $0 \leq a^+ \leq \|a\|e_0$. In particular $a^+ \in \text{im } P_0 = A_0$ (cf. [2 ; Cor.2.12]). Hence also $a^- = a^+ - a \in A_0$, and the proof is complete. \square

Corollary 1.5. With the assumptions and notation of Proposition 1.4, the functional calculus on (A_0, e_0) is given by $\varphi \rightarrow P_0(\varphi(a))$ where $\varphi \mapsto \varphi(a)$ is the functional calculus on (A, e) . For those continuous functions φ such that $\varphi(0) = 0$, the functional calculus on A_0 agrees with that of A , i.e. $P_0(\varphi(a)) = \varphi(a)$ for all $a \in A_0$.

Proof. Fix $a \in A_0$. The map $\varphi \mapsto P_0(\varphi(a))$ from bounded Borel functions on \mathbb{R} into A_0 will satisfy all the requirements for an extreme point preserving functional calculus (as specified in [2; Th.8.9]), and by the uniqueness of such a functional calculus, $\varphi \mapsto P_0(\varphi(a))$ must coincide with the functional calculus defined on (A_0, e_0) .

Now assume φ is continuous, with $\varphi(0) = 0$. Then there exists a sequence $\{\varphi_n\}$ of continuous functions, each vanishing in a neighbourhood of zero, such that $\varphi_n \rightarrow \varphi$ uniformly on $[-\|a\|, \|a\|]$. For each index n there exists a positive constant α_n such that $|\varphi_n(\lambda)| \leq \alpha_n |\lambda|$ for all $\lambda \in [-\|a\|, \|a\|]$. Since the functional calculus on A preserves order, we have the relations

$$-(a^+ + a^-) \leq \alpha_n^{-1} \varphi_n(a) \leq (a^+ + a^-).$$

By Proposition 1.4, a^+ and a^- are in A_0 , and therefore $\varphi_n(a) \in A_0$ for $n = 1, 2, \dots$. By norm continuity of the functional calculus, $\varphi_n(a) \rightarrow \varphi(a)$, so $\varphi(a) \in A_0$. Therefore $P_0(\varphi(a)) = \varphi(a)$ as claimed. \square

From Corollary 1.5 we immediately obtain the following:

Corollary 1.6. With the assumptions and notation of Proposition 1.4, the squaring operation $a \mapsto a^2$ for elements $a \in A_0$ is the same one whether calculated in A or in A_0 .

Our next corollary concerns the relativization of central P-projections.

Corollary 1.7. With the assumptions and notation of Proposition 1.4, let $P_0 \in \mathcal{P}$ be central. Then a P-projection $P \preceq P_0$ is central for A_0 iff it is central for A .

Proof. By [2; Prop.5.1] P is central for A (or A_0) iff $Pa \leq a$ for all $a \in A^+$ (respectively $A \in A_0^+$).

If $Pa \leq a$ for all $a \in A^+$, then clearly also $Pa_0 \leq a_0$ for all $a_0 \in A_0^+ \subseteq A^+$. Conversely, assume $Pa_0 \leq a_0$ for all $a_0 \in A_0^+$. For given $a \in A^+$ we have $P_0 a \leq a$ since P_0 is central. Hence

$$Pa = P(P_0 a) \leq P_0 a \leq a,$$

which completes the proof. \square

Definition. Suppose (A, e) and (V, K) are in spectral duality with $A = V^*$. Then we say A is a factor (with respect to the duality with V) if it does not admit any central P -projection other than 0 and I . Also we say that a subset $A_0 = \text{im } P$ where $P \in \mathcal{P}$, is a factor if A_0 is a factor in the above sense with respect to the duality with $V_0 = \text{im } P^*$.

Note that by Corollary 1.7 the subset $A_0 = \text{im } P_0$ where P_0 is central, will be a factor iff the only central P -projections $P \leq P_0$ are 0 and P_0 .

Proposition 1.8. Let (A, e) and (V, K) be in spectral duality with $A = V^*$. For each $\rho \in K$ there exists a smallest central projective unit $c(\rho)$ such that $\langle c(\rho), \rho \rangle = 1$; the corresponding P -projection P is the smallest P -projection such that $P^* \rho = \rho$; and the corresponding projective face is the smallest split face of K which contains ρ .

Proof. Recall first that for $\rho \in K$ and $P \in \mathcal{P}$ one has $\langle Pe, \rho \rangle = 1$ iff $P^* \rho = \rho$ [2; Lem.2.3].

Now, if c_1, c_2 are central projective units corresponding to central P -projections P_1, P_2 , and if $\langle c_1, \rho \rangle = \langle c_2, \rho \rangle = 1$, then

$$(P_1 \wedge P_2)^* \rho = (P_1 P_2)^* \rho = P_2^* P_1^* \rho = \rho ,$$

and so $\langle c_1 \wedge c_2, \rho \rangle = 1$. Therefore the set of central projective units c such that $\langle c, \rho \rangle = 1$, is directed downward. By [2 ; Lem.12.1] the pointwise limit $c(\rho)$ of this directed set exists and is a projective unit; by continuity $(P+P')c(\rho) = c(\rho)$ for all $P \in \mathcal{P}$, so $c(\rho)$ is central. Thus, $c(\rho)$ is the smallest central projective unit such that $\langle c(\rho), \rho \rangle = 1$.

By the introductory remark of this proof, the P -projection P corresponding to $c(\rho)$ is the smallest central P -projection such that $P^* \rho = \rho$. By definition the corresponding projective face is $F = K \cap \text{im} P^*$. Recalling that $K \cap \text{im} P^*$ is a split face iff $P \in \mathcal{P}$ is central [2 ; Prop.10.2], we conclude that F is the smallest split face containing ρ . \square

Definition. Let (A, e) and (V, K) be in spectral duality with $A = V^*$. For given $\rho \in K$ we denote by A_ρ the range of the P -projection corresponding to $c(\rho)$; thus $A_\rho = \text{im} P$ where $c(\rho) = Pe$. (By [2 ; Cor.2.12] A_ρ is also the order ideal of A generated by $c(\rho)$).

Proposition 1.9. Let (A, e) and (V, K) be in spectral duality with $A = V^*$. If ρ is an extreme point of K , then A_ρ is a factor.

Proof. Suppose for contradiction that there exists a central P -projection $Q \in \mathcal{P}$ such that $Q \leq P$, $Q \neq 0$, $Q \neq P$. Since Q is central, $Q + Q' = I$. Hence $(Q + Q')^* \rho = \rho$. Since ρ is

extreme, Q^*_ρ and Q'^*_ρ must both be multiples of ρ . Then necessarily $Q^*_\rho = 0$ or $Q'^*_\rho = 0$; otherwise Q^*_ρ would be a non-zero multiple of Q'^*_ρ which is impossible.

Now let $c_1 = Qe$ and $c_2 = c(\rho) - Qe$. By hypothesis $0 < c_1 < c(\rho)$ and c_1, c_2 are central projective units such that $c_1 + c_2 = c(\rho)$. If $Q^*_\rho = 0$, then $\langle c_1, \rho \rangle = \langle Qe, \rho \rangle = 0$, so $\langle c_2, \rho \rangle = \langle c(\rho) - c_1, \rho \rangle = 1$. If $Q'^*_\rho = 0$, then we similarly get $\langle c_1, \rho \rangle = 1$. In either case we have a contradiction with the minimality requirement defining $c(\rho)$. \square

We now turn to the study of minimal elements of \mathcal{U} .

Proposition 1.10. Let (A, e) and (V, K) be in spectral duality with $A = V^*$. If u is a minimal (non-zero) element of \mathcal{U} , then the corresponding P -projection has a 1-dimensional range, i.e. $\text{im } P \cong \mathbb{R}$ where $Pe = u$, and the corresponding projective face is a singleton, i.e. $F = \{\rho\}$ where $\rho \in K$ and $F = \{\sigma \in K \mid \langle \sigma, u \rangle = 1\}$. Moreover, the map $u \mapsto \rho$ is a 1-1 map of the minimal elements of \mathcal{U} onto the A -exposed points of K .

Proof. Let $u \in \mathcal{U}$ be minimal and let P be the corresponding P -projection, i.e. $u = Pe$. By [2; Cor.2.12], $\text{im } P$ is the order ideal of A generated by u . Therefore $\text{im } P \cong \mathbb{R}$ will follow if we can prove that $0 \leq a \leq u$ implies $a = \lambda_0 u$ for some $\lambda_0 \in \mathbb{R}^+$.

Now let $0 \leq a \leq u$, and let $\{e_\lambda\}$ be the spectral resolution of a . Since a is positive and $u \in \mathcal{U}$ is minimal, we have $e_\lambda = 0$ for $\lambda < 0$ and $e_0 = e - r(a) = e - u$. Hence also $e - e_\lambda \leq e - e_0 = u$ for $\lambda > 0$. From this it follows that there exists $\lambda_0 \in \mathbb{R}^+$ such that $e_\lambda = 0$ for $\lambda < \lambda_0$ and $e_\lambda = u$ for

$\lambda \geq \lambda_0$. Then $a = \lambda_0 u$, as desired.

From the above it also follows that $F = K \cap \text{im } P^*$ must consist of just one point $p \in K$.

Since F is a projective face, it is also an A -exposed face. Hence p is an A -exposed point of K . Clearly two different minimal elements of \mathcal{U} determine different A -exposed points in this way. Finally, if p is any A -exposed point, then $\{p\}$ is an A -exposed face; hence $\{p\}$ is a projective face. Clearly it is minimal among the (non-empty) projective faces; hence it corresponds to a minimal projective unit. \square

Remark. It follows from the above proposition that the minimal projective faces are (singletons consisting of) extreme points. It is not clear if the converse holds in this generality. But in § 3 we shall prove that it does hold in the JB-algebra setting.

Definitions. Let (A, e) and (V, K) be in spectral duality with $A = V^*$. Then a minimal non-zero projective unit in A will be called an atom. For each atom u we denote by \hat{u} the unique (A -exposed and extreme) point in the corresponding projective face. Thus \hat{u} is the unique point of K such that

$$(1.16) \quad \langle u, \hat{u} \rangle = 1.$$

If A is a factor containing at least one atom, then we say that A is a factor of type I.

§ 2. JB-algebras and spaces in spectral duality.

Following [3] we define a JB-algebra to be a Jordan algebra B over the reals with identity element e equipped with a complete norm such that for $a, b \in B$:

$$(2.1) \quad \|a \cdot b\| \leq \|a\| \cdot \|b\|$$

$$(2.2) \quad \|a^2\| \leq \|a\|^2$$

$$(2.3) \quad \|a^2\| \leq \|a^2 + b^2\|$$

Recall that if B is a JB-algebra, then the set B^2 of all squares in B is a proper convex cone organizing B to a (norm) complete order-unit space whose distinguished order-unit is the multiplicative identity and whose norm is the given one, and such that for $a \in B$:

$$(2.4) \quad -e \leq a \leq e \text{ implies } 0 \leq a^2 \leq e .$$

Conversely, if B is a complete order-unit space equipped with a Jordan product for which the distinguished order-unit acts as identity element and such that (2.4) is satisfied, then B is a JB-algebra in the order-unit norm [3; Th.2.1] .

In § 3 of [3] it is shown that one can associate to every JB-algebra B a monotone complete enveloping JB-algebra \tilde{B} , and in [20] it is proved that \tilde{B} can be identified with the bidual B^{**} equipped with the Arens product and the usual norm. In [20] there is also an investigation of JB-algebras which are dual spaces. Specifically, let A be a JB-algebra with identity element e such that $A = V^*$ for some Banach space V , and let K be the set of normal states for A . (A state ρ is a positive linear functional such that $\langle e, \rho \rangle = 1$, and it is said to be normal if whenever $\{a_\alpha\}$ is an increasing net in A with least

upper bound a , then $\langle a, \rho \rangle = \lim_{\alpha} \langle a_{\alpha}, \rho \rangle$. It is shown [20; Th.2.3] that the predual V is unique, in fact it can be identified with the space $\text{lin } K$ of all normal linear functionals in A^* .

Definition. In the sequel we will refer to JB-algebras which are dual spaces, as JBW-algebras. Also we will refer to the monotone complete enveloping JB-algebra $\tilde{B} \cong B^{**}$ of a JB-algebra B as the enveloping JBW-algebra of B .

It is natural to expect that a JBW-algebra and its predual are in spectral duality. We are now going to prove this result which will generalize Theorem 12.13 of [2].

Proposition 2.1. Let A be a JBW-algebra with identity e , predual V , and normal state space K . Then (A, e) and (V, K) are in spectral duality, and the map $a \mapsto a^2$ in the Jordan algebra A coincides with the squaring map defined by the functional calculus. The projective units in A are precisely the idempotent elements, and the P -projections are the maps U_p defined by Jordan triple products

$$(2.5) \quad U_p a = \{pap\} \quad \text{all } a \in A,$$

with p an idempotent in A ; the quasicomplementary P -projection for U_p is U_{e-p} .

Proof. In [20; Th.2.3] it is shown that under the natural embedding of V in $V^{**} = A^*$ the image of (V, K) is of the form $(\text{im } U_c^*, \tilde{K} \cap \text{im } U_c^*)$ where c is a central idempotent in $\tilde{A} = A^{**}$, and \tilde{K} is the state space of A . Now it follows from [2; Prop.2.14] that (V, K) is a base-norm space. In [20] it is also shown that each map U_a with $a \in A$ takes normal functionals to

normal functionals, and so is weakly (i.e. $\sigma(A,V)$ -) continuous. Given these results we complete the verification as in the proofs of Theorems 12.12 and 12.13 of [2]. \square

Corollary 2.2. The state space of a JB-algebra is a strongly spectral compact convex set.

Proof. Let B be an arbitrary JB-algebra and consider the order-unit space (A,e) where $A = \tilde{B}$ (the enveloping JBW-algebra) and the base-norm space (V,K) where $V = B^*$ and K is the state space of B . By Proposition 2.1 these two spaces are in spectral duality. (This also follows from [2; Th.12.13]). Hence K is a spectral compact convex set.

It follows from the uniqueness of the spectral functional calculus [2; Th.8.9] that the functional calculus defined by the spectral duality of (A,e) and (V,K) , will agree with that of [3; § 4]. In particular $B \cong A(K)$ is closed under functional calculus by continuous functions. Hence K is strongly spectral. \square

Having explained how JB-algebras give rise to spaces in spectral duality, we will now consider the converse problem of deriving Jordan structure from spectral duality under appropriate hypotheses.

Henceforth we assume that (A,e) and (V,K) are order-unit and base-norm spaces in spectral duality. Then there is a natural candidate for a Jordan product in A , namely

$$(2.6) \quad a \cdot b = \frac{1}{2} \{ (a+b)^2 - a^2 - b^2 \},$$

where the squares are defined by the functional calculus. This operation will coincide with the customary Jordan product

$a \circ b = \frac{1}{2}(ab+ba)$ when we specialize to operator algebras (cf. [2; Prop.11.4, Th.11.6]. But in the general case the product (2.6) can fail to be bilinear. However, it is proved in [2; Th.12.12] that if it is bilinear, then it organizes A to a JB-algebra. We will now proceed to give a necessary and sufficient condition for bilinearity of the operation (2.6).

First we recall that if A is a JB-algebra for the product (2.6) and if u is a projective unit corresponding to a P -projection P on A , then by Theorem 12.12 of [2] the corresponding multiplication operator $a \mapsto u \circ a$ is given by

$$(2.7) \quad u \circ a = \frac{1}{2}(a + Pa - P'a) .$$

This formula motivates the following general definition of the operator $T_u: A \rightarrow A$ associated with a given projective unit $u \in \mathcal{U}$ with $u = Pe$ for $P \in \mathcal{P}$:

$$(2.8) \quad T_u = \frac{1}{2}(I + P - P') .$$

For later references we state the following:

Lemma 2.3. Let (A, e) and (V, K) be an order-unit space and a base-norm space in spectral duality. If $u, v \in \mathcal{U}$ with $u = Pe$, $v = Qe$ for $P, Q \in \mathcal{P}$, then

$$(2.9) \quad [T_u, T_v]e = \frac{1}{4}[P - P', Q - Q']e = \frac{1}{2}([P, Q] + [P', Q'])e .$$

Proof. By linearity of commutators in each variable,

$$[T_u, T_v]e = \frac{1}{4}[I + P - P', I + Q - Q']e = \frac{1}{4}[P - P', Q - Q']e .$$

Again by linearity,

$$\frac{1}{4}[P - P', Q - Q']e = \frac{1}{4}([P, Q]e + [P', Q']e - [P, Q']e - [P', Q]e) .$$

Substituting $Qe = e - Q'e$, $Pe = e - P'e$ in the last two terms, we get

$$\frac{1}{4}[P-P', Q-Q'] = \frac{1}{2}([P, Q] + [P', Q'])e . \quad \square$$

Lemma 2.4. Let (A, e) and (V, K) be in spectral duality and assume that A is a JB-algebra for the product (2.6) Then for each pair $P, Q \in \mathcal{P}$:

$$(2.10) \quad [P, Q]e = [Q', P']e .$$

Proof. By commutativity of the Jordan product we obtain

$$T_u T_v e = T_u v = u \cdot v = v \cdot u = T_v u = T_v T_u e .$$

By Lemma 2.3, this completes the proof. \square

We will now prove that the condition (2.10) is sufficient as well as necessary in order that A be a JB-algebra.

Lemma 2.5. Let (A, e) and (V, K) be in spectral duality and assume that the condition (2.10) is satisfied for all pairs $P, Q \in \mathcal{P}$. Then the product (2.6) is bilinear on the space A_0 of all finite linear combinations of elements of \mathcal{U} ; moreover, for $u \in \mathcal{U}$ and $a, b \in A_0$:

$$(2.11) \quad T_u b = u \cdot b ,$$

$$(2.12) \quad \|a \cdot b\| \leq \|a\| \cdot \|b\| .$$

Proof. We first observe that if $a \in A_0$, say $a = \sum_{i=1}^n \lambda_i u_i$ with $u_1, \dots, u_n \in \mathcal{U}$ and if $v \in \mathcal{U}$, then by Lemma 2.3

$$\left(\sum_{i=1}^n \lambda_i T_{u_i} \right) v = \sum_{i=1}^n \lambda_i T_{u_i} T_v e = \sum_{i=1}^n \lambda_i T_v T_{u_i} e = T_v a .$$

Hence the value of $(\sum_{i=1}^n \lambda_i T_{u_i})v$ is independent of the particular representation $a = \sum_{i=1}^n \lambda_i u_i$. By linearity and continuity this result subsists with an arbitrary element $b \in A$ in place of v .

Thus, for every $a \in A_0$ there is a well defined operator $T_a : A \rightarrow A$ such that

$$(2.13) \quad T_a = \sum_{i=1}^n \lambda_i T_{u_i}$$

for any representation $a = \sum_{i=1}^n \lambda_i u_i$ with $u_1, \dots, u_n \in \mathcal{U}$.

Note that if $a = \sum_{i=1}^n \lambda_i u_i$ and $b = \sum_{j=1}^m \nu_j v_j$ with $u_1, \dots, u_n, v_1, \dots, v_m \in \mathcal{U}$, then

$$T_a b = \sum_{i=1}^n \lambda_i T_{u_i} \left(\sum_{j=1}^m \nu_j T_{v_j} e \right) = \sum_{j=1}^m \nu_j T_{v_j} \left(\sum_{i=1}^n \lambda_i T_{u_i} e \right) = T_b a.$$

Hence for all $a, b \in A_0$

$$(2.14) \quad T_a b = T_b a.$$

The next, and crucial, step is to prove that for all $a \in A_0$:

$$(2.15) \quad T_a a = a^2.$$

Observe first that if a has finite spectrum, i.e. if $a = \sum_{i=1}^n \lambda_i u_i$ with $u_i = P_i e$ for $P_i \in \mathcal{P}$ and with $u_i \perp u_j$ for $i \neq j$, then (2.15) holds. In fact,

$$\begin{aligned} T_a a &= \sum_{i=1}^n \lambda_i T_{u_i} \left(\sum_{j=1}^n \lambda_j u_j \right) = \sum_{i,j=1}^n \lambda_i \lambda_j T_{u_i} u_j \\ &= \sum_{i,j=1}^n \lambda_i \lambda_j P_i u_j = \sum_{i=1}^n \lambda_i^2 u_i = a^2. \end{aligned}$$

Now observe that by definition the map $a \mapsto T_a$ is linear

on A_0 , and so the map $(a,b) \mapsto T_a b$ is bilinear from $A_0 \times A_0$ into A . Thus for all $a,b \in A_0$

$$(2.16) \quad T_a b = \frac{1}{2}(T_{a+b}(a+b) - T_a a - T_b b) .$$

Now if a and b are compatible and have finite spectrum, then $a+b$ has finite spectrum (consider the abelian subspace $\cong C(X)$ generated by a and b), and so by (2.16) and the remarks above:

$$T_a b = \frac{1}{2}((a+b)^2 - a^2 - b^2) = a \cdot b .$$

For arbitrary $a \in A_0$, let $\{a_n\}$ be a sequence in the weakly closed abelian subspace $M(a)$ generated by a , such that $a_n \rightarrow a$ in norm and each a_n has finite spectrum. (Such a sequence exists by spectral theory). Then by (2.14) and norm continuity of the maps T_b with $b \in A_0$:

$$\begin{aligned} T_a a &= \lim_n T_a a_n = \lim_n \lim_m T_{a_n} a_m \\ &= \lim_n \lim_m a_n \cdot a_m = a^2 \end{aligned}$$

(where the last equality follows from continuity of the product on $M(a) \cong C(X)$). This establishes (2.15).

Combining (2.15) and (2.16) we now obtain for $a,b \in A_0$:

$$T_a b = \frac{1}{2}((a+b)^2 - a^2 - b^2) = a \cdot b .$$

This proves (2.11) as well as bilinearity of the product $a \cdot b$ on A_0 .

It remains to prove (2.12). To this end we assume $a,b \in A_0$ and $\|a\|, \|b\| \leq 1$. Recall also the general inequality

$$\|c-d\| \leq \max(\|c\|, \|d\|)$$

valid for any two positive elements c,d of an order-unit space,

and the equality

$$\|c^2\| = \|c\|^2$$

following from spectral theory [2; Prop.8.6, formula(8.25)].

Now

$$\begin{aligned} \|a \cdot b\| &= \frac{1}{4} \|(a+b)^2 - (a-b)^2\| \\ &\leq \frac{1}{4} \max(\|a+b\|^2, \|a-b\|^2) \leq 1, \end{aligned}$$

from which (2.12) follows. \square

Theorem 2.6. Let (A, e) be an order-unit space in spectral duality with a base-norm space (V, K) . Then A is a JB-algebra for the product

$$(2.17) \quad a \cdot b = \frac{1}{2} \{ (a+b)^2 - a^2 - b^2 \}$$

iff

$$(2.18) \quad [P, Q]e = [Q', P']e$$

for all pairs P, Q of P-projections on A .

Proof. The condition (2.18) is necessary by Lemma 2.4.

To prove that (2.18) is sufficient, it is enough to prove that the product $a \cdot b$ is bilinear [2; Th.12.12]. We have already shown that this product is bilinear on the dense subspace A_0 of A (Lemma 2.5). Hence we shall be through if we can prove that for two given sequences $\{a_n\}$ and $\{b_n\}$ in A_0 converging to $a \in A$ and $b \in A$ respectively, the product sequence $\{a_n \cdot b_n\}$ will converge to $a \cdot b$.

Thus, we assume $a_n, b_n \in A_0$ and $\|a - a_n\| \rightarrow 0$, $\|b - b_n\| \rightarrow 0$ as $n \rightarrow \infty$. By spectral theory there exists a sequence $\{a'_n\}$ in $M(a) \cap A_0$ such that $\|a - a'_n\| \rightarrow 0$ as $n \rightarrow \infty$. By continuity

of squaring in $M(a)$, we also have $(a'_n)^2 \rightarrow a^2$. Since $\|a_n - a'_n\| \rightarrow 0$, we obtain from Lemma 2.5

$$\begin{aligned} \|a_n^2 - (a'_n)^2\| &= \|a_n \circ a_n - a'_n \circ a'_n\| \\ &= \|a_n \circ (a_n - a'_n) + (a_n - a'_n) \circ a'_n\| \\ &\leq \|a_n\| \cdot \|a_n - a'_n\| + \|a_n - a'_n\| \cdot \|a'_n\|. \end{aligned}$$

The last expression tends to zero as $n \rightarrow \infty$. Hence $a_n^2 \rightarrow a^2$. Similarly we prove that $b_n^2 \rightarrow b^2$. Finally

$$\begin{aligned} \lim_n (a_n \cdot b_n) &= \lim \frac{1}{2} ((a_n + b_n)^2 - a_n^2 - b_n^2) \\ &= \frac{1}{2} ((a+b)^2 - a^2 - b^2) = a \cdot b, \end{aligned}$$

and the theorem is proved. \square

Corollary 2.7. For a given convex set K the following are equivalent:

- (i) K is affinely isomorphic to the normal state space of a JBW-algebra.
- (ii) K is spectral and $[P, Q]e = [Q', P']e$ for every pair P, Q of P -projections on $A = A^b(K)$.

Proof. If (i) holds, then (ii) will hold by Proposition 2.1 and Lemma 2.4.

Conversely, if (ii) holds, then K can be embedded in a linear space V in such a way that (V, K) becomes a base-norm space in spectral duality with (A, e) where $A = A^b(K) = V^*$. By Theorem 2.6 A can be equipped with a product making it a JB-algebra. Since A is a dual space, it is in fact a JBW-

algebra, and by [20; Th.2.3] the normal state space of A is affinely isomorphic to K . \square

Corollary 2.8. For a given compact convex set K the following are equivalent:

- (i) K is affinely isomorphic and homeomorphic to the state space of a JB-algebra equipped with the usual w^* -topology.
- (ii) K is strongly spectral and $[P, Q]e = [Q', P']e$ for every pair P, Q of P -projections on $A = A^b(K)$.

Proof. If (i) holds, then (ii) will hold by Corollary 2.2 and Lemma 2.4.

Conversely, if (ii) holds, then we consider the regular embedding of K into the base-norm space (V, K) where $V = A(K)^*$ [1; Ch II, §2]. Since K is spectral, (V, K) is in spectral duality with (A, e) where $A = A^b(K) \cong V^* = A(K)^{**}$. By Theorem 2.6, A can be equipped with a product making it a JB-algebra (and in fact even a JBW-algebra). The Jordan product of A is given by (2.17); hence the squaring map $a \mapsto a^2$ determined by this Jordan product, is the same as that determined by functional calculus. Since K is supposed to be strongly spectral, $A(K)$ is closed under the squaring map. Hence $B = A(K)$ must be a Jordan subalgebra of $A = A^b(K)$ containing e , and thus is a JB-algebra. By elementary properties of compact convex sets [1; Ch II, §2], the convex set K is linearly isomorphic and homeomorphic to the state space of $B = A(K)$ equipped with the w^* -topology. This completes the proof. \square

Remarks on the physical interpretation. We do not know of any natural geometric interpretation of the condition

$[P, Q]e = [Q', P']e$; our main concern in the sequel will be to replace it by requirements which are more geometric. However, the above condition does admit an interesting physical interpretation, which we will now briefly discuss.

Following [14] and [7] we may view each $\rho \in V^+$ as representing a beam of particles with intensity $\|\rho\| = \langle e, \rho \rangle$, so that the elements of K represent beams of unit intensity. Each pair Q, Q' of quasicomplementary P -projections represent complementary physical filters which serve to measure whether a particle does or does not possess a certain property "Q" . Thus, for an incoming beam ρ entering the Q -filter, the result will be an outgoing beam $Q^*\rho$ consisting of those particles which definitely do possess property Q . (In operative terms, this means that with probability 1 they will pass through any Q -filter placed after the original one). Similarly, for an incoming beam ρ entering the Q' -filter, the result will be an outgoing beam $Q'^*\rho$ consisting of those particles which definitely do not possess property Q . (With probability 1 these particles will be stopped by any Q -filter placed after the original Q' -filter). Note that $\|Q^*e\| = \langle e, Q^*\rho \rangle$ is the intensity of the beam emerging from the Q -filter; hence the ratio $\|Q^*\rho\|/\|\rho\|$ is the probability that a particle in the beam will pass through the Q -filter, i.e. have property Q . In particular, if $\rho \in K$, then $\|Q^*\rho\|$ is the probability that a particle in the beam has property Q .

Now consider a sequence of two measurements, first by either one of the filters P or P' , then by either one of the filters Q or Q' . Note that for example $Q^*P^*\rho$ represents that portion of an incoming beam $\rho \in K$ which will pass through the P -filter and then also through the Q -filter. Thus, $\|Q^*P^*\rho\|$ represents

the probability that a particle in the beam $\rho \in K$ will possess both properties P and Q , when the measurements are carried out in this order. Similarly, $\|Q'^*P^*\rho\|$ represents the probability that a particle in the beam $\rho \in K$ will possess property P but not Q , and so on. Starting from this, we can express the probability of any logical combination of the properties P and Q (keeping the assumption that the measurements of P are performed before the measurements of Q). In particular, we are interested in the "exclusive disjunction" (either one or the other, but not both). Here we obtain:

$$\begin{aligned}
 (2.19) \quad & \text{Prob}(P \text{ and } (\text{not } Q), \text{ or, } (\text{not } P) \text{ and } Q) \\
 &= \|Q'^*P^*\rho\| + \|Q^*P'^*\rho\| \\
 &= \langle e, Q'^*P^*\rho \rangle + \langle e, Q^*P'^*\rho \rangle \\
 &= \langle (PQ' + P'Q)e, \rho \rangle .
 \end{aligned}$$

Suppose we now reverse the order of the measurements. Then we get:

$$\begin{aligned}
 (2.20) \quad & \text{Prob}(Q \text{ and } (\text{not } P), \text{ or, } (\text{not } Q) \text{ and } P) \\
 &= \|P'^*Q^*\rho\| + \|P^*Q'^*\rho\| \\
 &= \langle e, P'^*Q^*\rho \rangle + \langle e, P^*Q'^*\rho \rangle \\
 &= \langle (QP' + Q'P)e, \rho \rangle .
 \end{aligned}$$

Finally observe that by replacing Q by Q' in the condition $[P, Q]e = [Q', P']e$, we can convert it into $[P, Q']e = [Q, P']e$, which is equivalent to

$$(2.21) \quad (PQ' + P'Q)e = (QP' + Q'P)e .$$

Comparison with (2.19) and (2.20) shows that the condition (2.21) is equivalent to the requirement that the probability of the exclusive disjunction of P and Q is independent of the order of the measurements of P and Q . Thus the key property (2.18) needed for Jordan structure is equivalent to this physical property. Note in particular that this requirement is satisfied in the usual model of quantum mechanics, in which K is the normal state space of the algebra of all bounded operators on a Hilbert space. (This follows from Lemma 2.4, but it can also be verified by elementary operations with Hilbert space operators).

§ 3. The ball and symmetry properties.

Our purpose in this section is to show that normal state spaces of JBW-algebras (and thus also state spaces of JB-algebras) possess two geometric properties which we call the Hilbert-ball property and the symmetry property, respectively. These properties are not generally enjoyed by spectral convex sets, and in a later section we will show that they do in fact characterize the state spaces of JB-algebras among the strongly spectral compact convex sets.

We begin by recalling some general notions relating to an arbitrary convex set K (in some linear space), and for brevity we denote by A the space $A^b(K)$ of all bounded affine functions on K . (This conforms with the notations used in §§ 1,2). For any subset E of K there exists a smallest face $F = \text{face}(E)$ which contains E (possibly $F = K$). A face F of K is said to be A -exposed if there exists a function $a \in A$ such that $a > 0$ on $K \setminus F$ and $a = 0$ on F . A point $\rho \in K$ is said to be A -exposed if $\{\rho\}$ is an A -exposed face. Clearly, every A -exposed point is extreme. A face F of K is said to be a split face if there exists a (necessarily unique) face F' such that every element of K can be written uniquely in the form $\lambda\rho + (1-\lambda)\sigma$ with $0 \leq \lambda \leq 1$, $\rho \in F$, $\sigma \in F'$. Note that if F is a split face with complementary face F' , then every extreme point of K is contained in $F \cup F'$. We say that two extreme points ρ, σ of K are separated by a split face if there exists a split face F such that $\rho \in F$ and $\sigma \in F'$. Finally we recall from § 1 that if K is spectral, then a face F of K will be A -exposed iff it is projective, and that every subset $E \subseteq K$ is

contained in a smallest projective (or A -exposed) face $F(E)$. (By [2; Prop.10.2] the split faces of K are precisely those faces which correspond to central P -projections.)

We shall need the following elementary result from general convexity theory.

Proposition 3.1. If two extreme points ρ, σ of a convex set K can be separated by a split face, then $\text{face}(\{\rho, \sigma\})$ is equal to the line segment $[\rho, \sigma]$. If in addition ρ and σ are A -exposed points, then $[\rho, \sigma]$ is an A -exposed face.

Proof. 1. The statement that $\text{face}(\{\rho, \sigma\}) = [\rho, \sigma]$, follows in a straightforward way from the definitions; we leave the details to the reader.

2. Assume now that ρ and σ are A -exposed, and let $a, b \in A$ be chosen such that $a > 0$ on $K \setminus \{\rho\}$, $\langle a, \rho \rangle = 0$ and $b > 0$ on $K \setminus \{\sigma\}$, $\langle b, \sigma \rangle = 0$. Also let F be a split face such that $\rho \in F$ and $\sigma \in F'$. For every $w \in K$, let $\lambda(w) \in [0, 1]$, $\varphi(w) \in F$ and $\psi(w) \in F'$ be (uniquely) determined by

$$(3.1) \quad w = \lambda(w)\varphi(w) + (1-\lambda(w))\psi(w).$$

Now define a function c on K by writing

$$(3.2) \quad \langle c, w \rangle = \lambda(w)\langle a, \varphi(w) \rangle + (1-\lambda(w))\langle b, \psi(w) \rangle.$$

Clearly c is bounded, and it is seen by a straightforward verification that c is affine. Hence $c \in A$. Clearly also $c > 0$ on $K \setminus [\rho, \sigma]$ and $c = 0$ on $[\rho, \sigma]$. This completes the proof. \square

By a Hilbert ball we shall understand the closed unit ball of some real Hilbert space H . The dimension of H can be

finite or infinite, and for convenience we also admit the "zero dimensional Hilbert ball", consisting of a single point.

Definition. A convex set K has the Hilbert ball property if $\text{face}(\{\rho, \sigma\})$ is an A -exposed face affinely isomorphic to a Hilbert ball for every pair ρ, σ of extreme points of K .

Note that if the two extreme points ρ, σ coincide, then $\text{face}(\{\rho, \sigma\}) = \text{face}\{\rho\} = \{\rho\}$; so if K has the Hilbert ball property, then every extreme point of K is A -exposed.

Note also that by Proposition 3.1, a convex set K will have the Hilbert ball property iff every extreme point is A -exposed and $\text{face}(\{\rho, \sigma\})$ is an A -exposed face affinely isomorphic to a Hilbert ball for every pair ρ, σ of distinct extreme points which can not be separated by a split face.

The convex sets to be studied in the sequel will all be spectral, and then the notions of " A -exposed face" and "projective face" will coincide, as noted above. Thus if a spectral convex set K has the Hilbert ball property, then

$$(3.3) \quad F(\{\rho, \sigma\}) = \text{face}(\{\rho, \sigma\})$$

for every pair ρ, σ of extreme points of K ; in particular

$$(3.4) \quad F(\rho) = \{\rho\}$$

for every single extreme point ρ of K .

We now proceed to prove that the normal state space of a JBW-algebra has the Hilbert ball property. But first we recall some preliminaries from the theory of JB-algebras.

For a JBW-algebra A the notion of "compatibility" of two elements $a, b \in A$ defined via the spectral duality of (A, e) with

its predual (V, K) , will coincide with "operator commutativity" of the spectral resolutions $\{e_\lambda^a\}$, $\{e_\mu^b\}$, as can be seen from [3; Lem.2.11]. Hence our definition of center for A (§ 1) is equivalent to the JB-algebra definition of center for A [3; §4]. We will use the term JB-factor to denote a JBW-algebra whose center is trivial. This definition is in agreement with the general definition of a factor for spaces in spectral duality (§ 1), but it is formally different from the definition of a JB-factor in [3], which does not involve the notion of a JBW-algebra. However, it can be seen from [20] that the two definitions are equivalent. By a minimal idempotent in a JBW-algebra A we mean a minimal element of the set of non-zero idempotents of A , or equivalently an atom (cf. § 1) defined in the spectral duality of (A, e) with its predual (V, K) . Recall from [3] that a JB-factor is said to be of type I if it contains a minimal idempotent (this is in agreement with the general definition given in § 1), and that it is said to be of type I_2 if the maximal cardinality of a set of orthogonal non-zero idempotents is 2.

The first step towards the Hilbert ball property will be to prove that every extreme point of the normal state space of a JBW-algebra A is A -exposed. We begin by showing this for the exceptional JB-algebra M_3^8 of all self-adjoint 3×3 -matrices over the Cayley numbers. This algebra is finite dimensional, so we do not have to worry about normality for states, and we can use the term "exposed point" without further specification.

Lemma 3.2. Every extreme point of the state space K of M_3^8 is exposed.

Proof. Consider the spectral duality of (A, e) with (V, K) where $A = M_3^8$ and $V = (M_3^8)^*$ (cf. Proposition 2.1). By a known result there exists in A an inner product making A^+ a self-dual cone, i.e. $(a|b) \geq 0$ for all $b \in A^+$ iff $a \in A^+$ (cf. [5; Ch.11, Satz 3.8]). Now $a \mapsto (|a)$ is seen to be an order isomorphism of A onto $A^* = V$. Therefore, if we can prove that every extreme ray of A^+ is exposed, then every extreme ray of V^+ will be exposed, and so every extreme point of K will be exposed.

But an extreme ray of A^+ is a closed face of A^+ , and therefore of the form $\text{im}^+ P$ for some P -projection P [2; Th.12.3]. By Proposition 1.4 of [2] $\text{im}^+ P$ is semi-exposed, and by finite dimensionality, exposed. This completes the proof. \square

Proposition 3.3. Every extreme point of the normal state space K of a JBW-algebra A is A -exposed.

Proof. Let ρ be an extreme point of K and let $c(\rho)$ be the smallest central idempotent of A satisfying $\langle c(\rho), \rho \rangle = 1$ (cf. Proposition 1.8). Also let P_ρ be the corresponding P -projection, i.e. $c(\rho) = P_\rho e$, and let F_ρ be the corresponding projective face, i.e. $F_\rho = \{\sigma \in K \mid \langle c(\rho), \sigma \rangle = 1\}$. Note that $P_\rho = U_{c(\rho)}$ by Proposition 2.1. To show that ρ is A -exposed, we will relativize to the spectral duality of $(A_\rho, c(\rho))$ and (V_ρ, F_ρ) where $A_\rho = \text{im} P_\rho$ and $V_\rho = \text{im} P_\rho^*$ (cf. Proposition 1.4). Specifically, we will show that $\{\rho\}$ is an A_ρ -exposed, or equivalently a projective, face of F_ρ .

By Proposition 1.8 (or by [3; Lem.5.5]) A_ρ is a factor. Therefore by [3; Th.8.6] A_ρ is either isomorphic to M_3^8 or to a Jordan algebra of operators on a Hilbert space (a "JC-algebra").

In the former case we are done by Lemma 3.2. In the latter case the argument in the proof of Theorem 8.7 in [3] shows that the projective face generated by ρ is minimal. It then follows from Proposition 1.10 that this face consist of ρ only. Thus, $\{\rho\}$ is a projective face of F_ρ , as desired. \square

Next we shall prove a series of lemmas needed for the study of $\text{face}(\{\rho, \sigma\})$ for two extreme points ρ, σ of the normal state space of a JBW-algebra.

Lemma 3.4. Let (A, e) be a JBW-algebra in spectral duality with its predual (V, K) (cf. Proposition 2.1). Let $u, v \in \mathcal{U}$ be arbitrary. Then there exists $s \in A$ with $s^2 = e$ such that

$$(3.5) \quad U_s(u \vee v - u) = v - u \wedge v .$$

Proof. By Lemma 6.2 of [3] there exists $s \in A$ with $s^2 = e$ such that

$$(3.6) \quad U_s\{u'vu'\} = \{vu'v\} ,$$

where $u' = e - u$. By the proof of Lemma 6.3 of [3] we also have

$$(3.7) \quad U_s r(\{u'vu'\}) = r(\{vu'v\}) .$$

By Lemma 1.2 and equation (1.10):

$$r(\{u'vu'\}) = (v \vee u) \wedge u' = (v \vee u) - u$$

and

$$r(\{vu'v\}) = (u' \vee v') \wedge v = (u \wedge v)' \wedge v = v - (u \wedge v) ,$$

which completes the proof. \square

The following corollary will not be needed in the sequel,

but seems of interest in its own right. Let L be a lattice and $a, b \in L$; one says (a, b) is a modular pair (written $(a, b)M$) if for all $x \in L$ with $a \wedge b \leq x \leq b$ there holds $x = (x \vee a) \wedge b$. A lattice is semimodular if the relation M is symmetric, i.e. if $(a, b)M$ implies $(b, a)M$. (For background see [4] or [13]). Topping [22] has shown that the projection lattice of a von Neumann algebra is semimodular. We will now generalize this to JBW-algebras.

Corollary 3.5. The lattice \mathcal{U} of idempotents (= projective units) in a JBW-algebra A is semimodular.

Proof. We know that \mathcal{U} is an orthomodular lattice (cf. [2; Th.4.5] or [3; Prop.4.9]). By Corollary 36.14 and Theorem 29.8 of [13], it now suffices to prove that whenever $u, v \in \mathcal{U}$ and $u \vee v = e$, $u \wedge v = 0$, there exists a lattice automorphism of \mathcal{U} taking u to v' and v to u' .

If $s \in A$ and $s^2 = e$, then the map $U_s: a \mapsto \{sas\}$ satisfies $U_s^2 = I$ and it maps idempotents into idempotents; by [3; Prop.2.7] it is order preserving so it is a lattice automorphism of \mathcal{U} .

Now if $u \vee v = e$ and $u \wedge v = 0$, then Lemma 3.4 gives $U_s(e-u) = v$ for some $s \in A$ with $s^2 = e$. Thus $U_s u = e - v = v'$ and $U_s v = u'$, which completes the proof that \mathcal{U} is semimodular. \square

Lemma 3.6. If u is a minimal idempotent of a JBW-algebra and v is an arbitrary idempotent, then $u \vee v - v$ is either a minimal idempotent or zero.

Proof. By Lemma 3.4 there exists $s \in A$ with $s^2 = e$,

such that U_s exchanges $u \vee v - v$ and $v - u \wedge v$. By minimality of v , the latter either equals v or is zero. Now note that since U_s is a Jordan automorphism (cf. [10; p.60] or use Macdonald's Theorem [10; p.41]), then U_s maps minimal idempotents to minimal idempotents. This proves the Lemma. \square

Lemma 3.7. Let u, v be distinct minimal idempotents of a JBW-algebra. If w is an idempotent different from 0 and $u \vee v$ such that $w \leq u \vee v$, then w is also minimal.

Proof. Define $w^\# = u \vee v - w$. Then $u \leq u \vee w^\# \leq u \vee v$; we claim one of these relations is actually an equality. For if both inequalities are strict, then we also have the strict inequalities

$$0 < u \vee w^\# - u < u \vee v - u,$$

which contradicts the fact that the last term is a minimal idempotent by Lemma 3.6. Thus, either $u = u \vee w^\#$ or $u \vee w^\# = u \vee v$.

In the former case, minimality of u forces $u = w^\#$; so

$$w = u \vee v - w^\# = u \vee v - u,$$

which is minimal by Lemma 3.6.

In the latter case

$$w = u \vee v - w^\# = u \vee w^\# - w^\#,$$

which is again minimal by Lemma 3.6. \square

Remark. We note for future use that the conclusion of Lemma 3.7 will hold with the same proof in the more general case where u, v are atoms and w is a general projective unit for

an order-unit space (A, e) in spectral duality with a base-norm space (V, K) , provided that the conclusion of Lemma 3.6 holds.

Lemma 3.8. If A is a JB-factor and u is a non-zero idempotent in A , then $\{uAu\}$ is a JB-factor.

Proof. Note first that $\{uAu\}$ is a JBW-algebra, as can be seen from [3 ; Prop. 4.11].

Now suppose that a is a non-trivial idempotent which is central in $\{uAu\}$, and let $b = u - a$. By Lemmas 6.4 and 6.7 of [3] there exist non-zero idempotents $a_1 \leq a$, $b_1 \leq b$ and $s \in A$ with $s^2 = e$ such that $\{sa_1s\} = b_1$. Let $t = \{usu\}$; then since $a_1, b_1 \in \{uAu\}$

$$\{ta_1t\} = \{u\{s\{ua_1u\}s\}u\} = b_1 .$$

But by positivity of the maps U_t and U_b [3 ; Prop.2.7] we now get

$$b_1 = \{bb_1b\} = \{b\{ta_1t\}b\} \leq \{b\{tat\}b\} ,$$

and since $a, b, t \in \{uAu\}$ are compatible by centrality of a and b , this gives (cf.[3 ; Lem.2.11])

$$b_1 \leq U_b U_t a = U_t U_b a = 0 ,$$

a contradiction. This completes the proof that there is no non-trivial central idempotent in $\{uAu\}$, so $\{uAu\}$ is a JB-factor. \square

Lemma 3.9. If u, v are distinct minimal idempotents in a JB-factor A , then $A_{u, v} = \{(uvv)A(uvv)\}$ is a type I_2 JB-factor.

Proof. By Lemma 3.8, $A_{u,v}$ is a JB-factor. Since $A_{u,v}$ contains the minimal idempotents u and v , it must be of type I. Note also that the maximal number of elements in a set of orthogonal idempotents is at least 2, since u and $u \vee v - u$ are orthogonal and both non-zero. On the other hand, suppose that w_1, w_2, w_3 were three non-zero orthogonal idempotents in $A_{u,v}$. Then $w_1 + w_2$ would be a non-minimal idempotent under $u \vee v$, contrary to Lemma 3.7. Thus, there can not be any set of more than 2 orthogonal non-zero idempotents in $A_{u,v}$; so $A_{u,v}$ is an I_2 -factor. \square

Lemma 3.10. If A is a JB-factor of type I_2 , then all states of A are normal and the state space of A is affinely isomorphic to a Hilbert ball.

Proof. Let A be any JB-factor of type I_2 , and note that by Proposition 7.1 of [3] A will be an (abstract) spin factor in the sense of Topping [21], i.e. A can be equipped with an inner product $(a|b)$ which makes it a real Hilbert space in such a way that e is a unit vector and for every pair a, b of elements of $N = \{e\}^\perp$ one has

$$(3.8) \quad a \cdot b = (a|b)e.$$

In particular $\|a\|^2 = \|a^2\| = |(a|a)|$, so the Hilbert norm $\|a\| = \sqrt{(a|a)}$ will coincide with the given JB-algebra norm $\|a\|$ for $a \in N$. Clearly also the two norms will coincide on Re . By Lemmas 3 and 4 of [21] the following inequalities will hold for a general element $a \in A$

$$(3.9) \quad \|a\| \leq \|a\| \leq \sqrt{2} \|a\|.$$

Therefore A is linearly homeomorphic to a Hilbert space, and so is reflexive. Hence the predual of A will coincide with its dual, and all states must be normal.

We will prove that the closed unit ball of the subspace N of A provided with the Hilbert norm $\|a\|$, is affinely isomorphic to the state space of A . To this end we first note that by Lemma 1 of [21] a general element $a + \alpha e \in A$, with $a \in N$ and $\alpha \in \mathbb{R}$, is positive iff

$$(3.10) \quad \alpha \geq \|a\|.$$

For every given $b \in N$ we define a linear functional ρ_b on A by

$$(3.11) \quad \rho_b(a + \alpha e) = (a|b) + \alpha,$$

where $a \in N$ and $\alpha \in \mathbb{R}$.

Clearly $b \mapsto \rho_b$ is an affine map of N into A^* . If $\|b\| \leq 1$, then for every $a + \alpha e \geq 0$ with $a \in N$ and $\alpha \in \mathbb{R}$ we get by (3.10) and (3.11):

$$\rho_b(a + \alpha e) \geq \alpha - |(a|b)| \geq \alpha - \|a\| \geq 0.$$

Hence $\rho_b \geq 0$ when $\|b\| \leq 1$. By the definition (3.11), $\rho_b(e) = 1$ for all $b \in N$. Thus, $b \mapsto \rho_b$ is an affine map of the closed unit ball of N into the state space of A .

Now let b_1, b_2 be two distinct vectors in N . Letting $a = b_1 - b_2 \neq 0$, we have by (3.11)

$$\rho_{b_1}(a) - \rho_{b_2}(a) = (a|b_1 - b_2) = \|b_1 - b_2\|^2 \neq 0.$$

Hence the map $b \mapsto \rho_b$ is 1-1.

Finally we consider an arbitrary state ρ of A . Since

the two norms on A coincide on N , the restriction of ρ to N will have norm at most 1 with respect to the Hilbert norm. Hence there exists $b \in N$ with $\|b\| \leq 1$ such that $\rho(a) = (a|b)$ for all $a \in N$. By linearity

$$\rho(a + \alpha e) = (a|b) + \alpha \rho(e) = (a|b) + \alpha$$

for arbitrary $a \in N$ and $\alpha \in \mathbb{R}$. Hence $\rho = \rho_b$. This proves that the map $b \mapsto \rho_b$ maps the unit ball of N onto the state space of A , and we are done. \square

Theorem 3.11. The normal state space of a JBW-algebra has the Hilbert ball property.

Proof. Let A be any JBW-algebra with normal state space K . By Proposition 3.3 every extreme point of K is A -exposed. We consider an arbitrary pair ρ, σ of extreme points of K which can not be separated by a split face, and we will prove that $\text{face}(\{\rho, \sigma\})$ is an A -exposed face affinely isomorphic to a Hilbert ball. By previous remarks this will prove our theorem.

Let R be the P -projection, and w the projective unit, corresponding to the projective face $F(\{\rho, \sigma\})$ generated by ρ and σ . Since $\{\rho\}$ and $\{\sigma\}$ are A -exposed faces, they will also be projective faces. We denote the corresponding (minimal) projective units by u and v , respectively. In the lattice \mathcal{F} of projective faces of K , we have $F(\{\rho, \sigma\}) = \{\rho\} \vee \{\sigma\}$. Hence in the lattice \mathcal{U} of projective units, we have $w = u \vee v$. Therefore R is the P -projection corresponding to $u \vee v$, and by Proposition 2.1, $R = U_{u \vee v}$. Thus, for $a \in A$

$$(3.12) \quad Ra = \{(u \vee v)a(u \vee v)\}.$$

We write $A_{u,v} = \text{im } R$, and we note that $A_{u,v}$ is a JBW-algebra by [3 ; Prop.4.11]. By Proposition 1.4 ($\text{im } R^*$, $F(\{\rho, \sigma\})$) will be the predual of $(A_{u,v}, u \vee v)$. Thus $F(\{\rho, \sigma\})$ will be the normal state space of $A_{u,v}$.

Now, let $c(\rho)$ be the smallest central projective unit of A such that $\langle c(\rho), \rho \rangle = 1$, and recall that the corresponding projective face G is the smallest split face containing ρ (Proposition 1.8). Since the two extreme points ρ and σ can not be separated by a split face, then also $\sigma \in G$. It follows that $F(\{\rho, \sigma\}) \subseteq G$, and by passage to projective units, $u \vee v \leq c(\rho)$.

Let Q be the (central) P-projection corresponding to $c(\rho)$, let $A_{c(\rho)} = \text{im } Q$, and note that $u \in A_{c(\rho)}$ and $v \in A_{c(\rho)}$. (Recall that by [2 ; Cor.2.12], $\text{im } Q$ is the order ideal of A generated by $c(\rho)$). By Lemma 5.5 of [3], $A_{c(\rho)}$ is a JB-factor; and since it contains the minimal idempotents u, v , it must be of type I.

Next we note that $A_{u,v} = RA = RA_{c(\rho)}$ since $A_{u,v} \subseteq A_{c(\rho)}$. Then it follows from Lemma 3.9 and formula (3.12) that $A_{u,v}$ is an I_2 -factor. By Lemma 3.10 the normal state space $F(\{\rho, \sigma\})$ of $A_{u,v}$ must be a Hilbert ball.

Finally we note that the face generated by any pair of distinct extreme points of a Hilbert ball, is the entire ball. Hence $\text{face}(\{\rho, \sigma\}) = F(\{\rho, \sigma\})$. Thus, $\text{face}(\{\rho, \sigma\})$ is a projective, hence A -exposed, face of K affinely isomorphic to a Hilbert ball. \square

Corollary 3.12. The state space of a JB-algebra has the Hilbert ball property.

Proof. The state space of a JB-algebra is the normal state

space of its enveloping JBW-algebra (cf. [20]), so Theorem 3.11 applies. \square

We now turn to discuss a second geometric property: symmetry. First we define a reflection of a convex set K (in some linear space) to be an affine automorphism φ of K which is involutory (i.e. $\varphi^2 = \text{id}$). To justify the term "reflection", note that such a φ has a non-empty set of fixed points $K_0 = \{\frac{1}{2}(\rho + \varphi(\rho)) \mid \rho \in K\}$, and that for each $\rho \in K$ the image point $\varphi(\rho)$ is obtained by reflecting the line segment $[\rho, \varphi(\rho)]$ about its mid-point $\frac{1}{2}(\rho + \varphi(\rho)) \in K_0$.

Definition. A convex set K is symmetric with respect to a convex subset K_0 if there exists a reflection of K whose set of fixed points is precisely K_0 .

Lemma 3.13. Let K be a convex set embedded in a linear space V in such a way that (V, K) is a base-norm space in separating duality with an order unit space (A, e) where $A \cong V^*$. If F is a projective face of K and P is the corresponding P -projection on A , then the following are equivalent:

- (i) K is symmetric with respect to $\text{co}(F \cup F^\#)$
- (ii) $2P + 2P' - I \geq 0$.

If these equivalent conditions hold, then $\varphi = (2P + 2P' - I)^*|_K$ is the unique reflection of K with $\text{co}(F \cup F^\#)$ as its set of fixed points.

Proof. Assume first that (i) holds, and let φ be a reflection of K with $\text{co}(F \cup F^\#)$ as its set of extreme points. Define $\psi = \frac{1}{2}(\varphi + \text{id})$, and note that $\psi(K) \subset K$ and $\psi^2 = \psi$.

Furthermore, for $\rho \in K$ it is easily verified that $\psi(\rho) = \rho$ iff $\varphi(\rho) = \rho$, so $\psi(K) = \text{co}(F \cup F^\#)$. Thus ψ is an affine retraction of K onto $\text{co}(F \cup F^\#)$. By [2 ; Th.3.8] there exists exactly one affine retraction of K onto $\text{co}(F \cup F^\#)$, namely $(P + P')^*|_K$. Thus we have $\psi = (P + P')^*|_K$, which implies $\varphi = (2P + 2P' - I)^*|_K$. In particular $(2P + 2P' - I)^*$ leaves the base K of the cone V^+ invariant. Hence $2P + 2P' - I$ leaves the cone A^+ invariant, so (ii) is proven. In addition we have proven the uniqueness statement of the lemma.

Assume next that (ii) holds. Then since $2P + 2P' - I \geq 0$ and $(2P + 2P' - I)e = e$, the dual map $(2P + 2P' - I)^*$ will leave invariant not only the cone V^+ but also its base K . Since $(2P + 2P' - I)^2 = I$, the map $(2P + 2P' - I)^*|_K$ is a reflection of K . Furthermore $(2P + 2P' - I)^*\rho = \rho$ iff $(P + P')^*\rho = \rho$, which by [2 ; Th.3.8] is equivalent to $\rho \in \text{co}(F \cup F^\#)$. This shows that K is symmetric with respect to $\text{co}(F \cup F^\#)$, which completes the proof. \square

Lemma 3.13 will apply in the special context of spectral convex sets.

Definition. A spectral convex set K is said to be symmetric if it is symmetric with respect to $\text{co}(F \cup F^\#)$ for every projective face F , or what is equivalent (by Lemma 3.13), if:

$$(3.13) \quad 2P + 2P' - I \geq 0 \quad \text{for all } P \in \mathcal{P}.$$

Theorem 3.14. The normal state space of a JBW-algebra is symmetric.

Proof. Let A be a JBW-algebra with normal state space K .

We will first verify the following identity valid for any idempotent $q \in A$ with $q' = e - q$:

$$(3.14) \quad \{(q-q')a(q-q')\} = 2\{q a q\} + 2\{q' a q'\} - a .$$

By definition, $\{bab\} = 2b \cdot (b \cdot a) - b^2 \cdot a$ for any $b \in A$. Applying this to $b = q - q' = 2q - e$, we find

$$(3.15) \quad \{(q-q')a(q-q')\} = 4\{q a q\} - 4q \cdot a + a .$$

Interchanging the roles of q and q' , we get

$$(3.16) \quad \{(q-q')a(q-q')\} = 4\{q' a q'\} - 4q' \cdot a + a .$$

Addition of (3.15) and (3.16) gives (3.14).

Now suppose P is any P -projection on A . Then by Proposition 2.1, $P = U_q$, $P' = U_{q'}$, for some idempotent $q \in A$. Thus (3.14) gives the following equality valid for each $a \in A$:

$$(2P + 2P' - I)a = (2U_q + 2U_{q'} - I)a = \{(q-q')a(q-q')\} .$$

By Proposition 2.7 of [3] the maps $U_b : a \mapsto \{bab\}$ are positive for all b , and so we conclude $2P + 2P' - I \geq 0$, which proves symmetry of K . \square

Passing from a JB-algebra to its enveloping JBW-algebra, we also obtain:

Corollary 3.15. The state space of a JB-algebra is symmetric.

Remark. In the proof above $s = q - q'$ satisfies $s^2 = e$. Thus each map $2P + 2P' - I$ in the JB-algebra context is of the form U_s for some element s such that $s^2 = e$. In the operator

algebra context the map U_s is conjugation by the self-adjoint unitary s . Thus symmetry of the state space for a C^* -algebra can be viewed as a consequence of the fact that conjugation by a self-adjoint unitary is an involutory $*$ -automorphism, and thus induces a reflection of the state space.

§ 4. Consequences of the ball and symmetry properties.

In this section we will derive as consequences of the ball and symmetry properties certain properties of the extreme points; these properties each have a natural physical interpretation, and they will be useful in the sequel. In fact, we will show in a later section that they can replace the ball and symmetry properties as necessary and sufficient conditions for a strongly spectral compact convex set to be the state space of a JB-algebra.

Our setting in this section will be that of a spectral convex set K which we think of as embedded in a linear space V in such a way that (V, K) is a base-norm space in spectral duality with an order unit-space (A, e) where $A = V^*$, and we will use terminology from spectral theory with reference to this duality.

The extreme rays of V^+ are the sets $R^+\rho$ with ρ an extreme point of K . Thus, to say that a map P^* (with $P \in \mathcal{P}$) preserves extreme rays, means that $P^*\rho$ is a multiple of an extreme point for every extreme point ρ of K . (Here we allow the possibility that $P^*\rho = 0$).

Proposition 4.1. If a spectral convex set K is symmetric and has the Hilbert ball property, then P^* preserves extreme rays for every $P \in \mathcal{P}$.

Proof. Let $\rho \in K$ be an arbitrary extreme point. We can, and shall, assume $(P + P')^*\rho \neq \rho$; for otherwise $0 \leq P^*\rho \leq \rho$, and then $P^*\rho$ would be a multiple of the extreme point ρ .

By the symmetry property, the map $(2P + 2P' - I)^*$ acts as an automorphism of K . Hence $\omega = (2P + 2P' - I)^*\rho$ is an extreme point of K . By the ball property, $\text{face}(\{\rho, \omega\})$ is affinely isomorphic to a Hilbert ball.

Now let $\lambda = \|P^* \rho\| \neq 0$, and observe that $1 - \lambda = \|P'^* \rho\| \neq 0$. Define also $\sigma = \|P^* \rho\|^{-1} P^* \rho$ and $\tau = \|P'^* \rho\|^{-1} P'^* \rho$. Then

$$\frac{1}{2}(\rho + \omega) = (P + P')^* \rho = \lambda \sigma + (1 - \lambda) \tau.$$

Now it follows from the definition of a face that $\sigma, \tau \in \text{face}(\{\rho, \omega\})$.

The points σ and τ are contained in disjoint faces $K \cap \text{im } P^*$ and $K \cap \text{im } P'^*$; therefore the line segment $[\sigma, \tau]$ can not be extended beyond σ or τ within K . Thus, σ and τ must be boundary points, and then also extreme points of the ball $\text{face}(\{\rho, \omega\})$. It follows that $\sigma = \|P^* \rho\|^{-1} P^* \rho$ is an extreme point of the given convex set K , and the proof is complete. \square

In our next proposition we will introduce two properties which are consequences of the preservation of extreme rays by P^* for $P \in \mathcal{P}$. In fact, under the assumption that every extreme point of K is exposed, it is not hard to verify that each one of them is equivalent with the preservation of extreme rays. However, this will not be needed in the sequel, so we omit the proof.

Proposition 4.2. If K is a spectral convex set such that P^* preserves extreme rays for every $P \in \mathcal{P}$, then:

- (i) For each atom $u \in \mathcal{U}$ and each $v \in \mathcal{U}$ $(u \vee v') \wedge v$ is either an atom or zero.
- (ii) Each $P \in \mathcal{P}$ maps atoms to multiples of atoms.

Proof. 1.) Let Q be the P -projection and G the projective face corresponding to $v \in \mathcal{U}$, and recall that $\{\hat{u}\}$ is the projective face corresponding to the atom $u \in \mathcal{U}$. By

Corollary 1.3

$$(4.1) \quad F(Q^*\hat{u}) = (\{\hat{u}\} \vee G^\#) \cap G .$$

By hypothesis $Q^*\hat{u}$ is either zero or a multiple of an extreme point. In the latter case $\{\|Q^*\hat{u}\|^{-1}Q^*\hat{u}\}$ is an A -exposed, hence projective, face of K . Therefore $\|Q^*\hat{u}\|^{-1}Q^*\hat{u} = \hat{w}$ for some atom $w \in \mathcal{U}$. Now (4.1) can be rewritten in the form $\{\hat{w}\} = (\{\hat{u}\} \vee G^\#) \cap G$. By the isomorphism of the lattice \mathcal{U} of projective units and the lattice \mathcal{F} of projective faces, this gives the equality $w = (u \vee v') \wedge v$, which completes the proof of (i).

2.) Let $P \in \mathcal{P}$ and let $u \in \mathcal{U}$ be an atom. By Lemma 1.2

$$r(Pu) = (u \vee P'e) \wedge Pe ,$$

and by statement (i) the right term is either zero or an atom. Therefore Pu must be a multiple of an atom. \square

Corollary 4.3. Let K be a spectral convex set such that P^* preserves extreme rays for every $P \in \mathcal{P}$. Now if $u, v \in \mathcal{U}$ are distinct atoms and $w \in \mathcal{U}$ satisfies $w \leq u \vee v$, then w is either an atom or else $w = 0$ or $w = u \vee v$.

Proof. By statement (i) of Proposition 4.2 $(u \vee v) \wedge v'$ is either an atom or zero. By (1.10) $(u \vee v) \wedge v' = (u \vee v) - v$, and therefore the conclusion of Lemma 3.6 holds in the lattice \mathcal{U} of projective units in A . Now the Corollary follows from the remark after Lemma 3.7. \square

Remarks. As was first observed by Pool in a related context [16], the statement (i) of Proposition 4.2 is closely related to

the notion of semimodularity of the lattice \mathcal{U} . It is well known that for every orthomodular lattice semimodularity implies (i), and it can be proved that the converse implication holds if there are "sufficiently many" atoms, i.e. if every element of the lattice is l.u.b. of atoms. (See [17] and [13; Th.30.2, Cor.7.7]).

Proposition 4.4. If K is a spectral convex set with the Hilbert ball property, then for every pair of atoms $u, v \in \mathcal{U}$:

$$(4.2) \quad \langle u, \hat{v} \rangle = \langle v, \hat{u} \rangle$$

Proof. By hypothesis $F = \text{face}(\{\hat{u}, \hat{v}\})$ is an A -exposed face affinely isomorphic to a Hilbert ball. Being A -exposed, the face F is also projective, therefore of the form $F = K \cap \text{im} Q^*$ for $Q \in \mathcal{P}$. Since $\hat{u}, \hat{v} \in F$, we can pass to the corresponding projective units and obtain $u, v \leq Qe$. Now by Proposition 1.4, u and v will determine projective units in the relativized spectral duality of $(\text{im} Q, Qe)$ and $(\text{im} Q^*, F)$. Thus the restriction of u and v to F will be affine functions with values in $[0, 1]$; the former with the values 1 at \hat{u} and 0 at its antipodal point, and the latter with the values 1 at \hat{v} and zero at its antipodal point.

Denoting by π the center of the ball F and by α the angle between $\hat{u} - \pi$ and $\hat{v} - \pi$ and using some elementary plane geometry, we obtain (see fig. 1) :

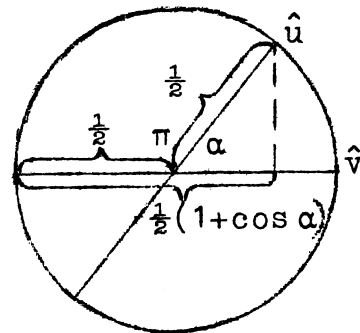


Fig. 1.

$$\langle u, \hat{v} \rangle = \langle v, \hat{u} \rangle = \frac{1}{2}(1 + \cos \alpha) .$$

Note that this proof is valid also when F is one-dimensional:
Then $\alpha = 180^\circ$ and $\langle u, \hat{v} \rangle = \langle v, \hat{u} \rangle = 0$. \square

The properties established in Propositions 4.1 and 4.4, will always be used in connection with the property that all extreme points of K are A -exposed. It will be convenient to be able to refer to them by a single name; since they are all related to the extreme points of K , we give the following:

Definition. A spectral convex set K has the pure state properties if:

(4.3) Every extreme point of K is A -exposed.

(4.4) P^* preserves extreme rays for every $P \in \mathcal{P}$.

(4.5) $\langle u, \hat{v} \rangle = \langle v, \hat{u} \rangle$ for every pair of atoms
 $u, v \in \mathcal{U}$.

Remarks on the physical interpretation of the pure state properties. As in § 2 we may view each $\rho \in V^+$ as representing a beam of particles with intensity $\|\rho\| = \langle e, \rho \rangle$, and each $P \in \mathcal{P}$ as representing a filter transforming any given beam $\rho \in V^+$ to a new beam $P^* \rho$ of intensity at most equal to that of the given beam. (By a fundamental property of P -projections, the intensity remains undiminished only when the filter is neutral to the beam, cf. [2 ; § 2]).

Property (4.3) states that for an arbitrary beam of particles in a pure state, i.e. for an arbitrary extreme point ρ of K , there exists a filter $P \in \mathcal{P}$ which transforms every incoming beam to (a multiple of) the given beam, i.e. $P^*(V^+) = \mathbb{R}^+ \rho$. (This may be expressed by saying that the filter P "prepares" ρ).

Property (4.4) states that each filter $P \in \mathcal{P}$ transforms pure states to (scalar multiples of) pure states. (The scalar factor represents the decrease in intensity, and is strictly less than 1 unless $P^* \rho = \rho$, as noted above).

Property (4.5) is a statement of a symmetric relationship between the "transition probabilities" connecting pure states. For a more detailed discussion of these and related properties see [16], [8], [14].

§ 5. Ellipticity.

In this section we will introduce and study a new property which is relevant in the context of spectral convex sets. This property, which we call ellipticity, is closely related to the notion of "facial homogeneity" defined by Connes [6] in a somewhat different context. We will prove in this section that the state spaces of JB-algebras are elliptic. The notion of ellipticity will play an important role in the development leading up to the main results in § 7, but it is not (as far as we know) sufficient as an axiom to characterize state spaces of JB-algebras. In fact, it is an open problem whether ellipticity alone will suffice to guarantee that a strongly spectral compact convex set is the state space of a JB-algebra.

We begin by a lemma describing the normalized orbits of one-parameter groups $t \mapsto \exp t(P-P')^* \rho$ determined by P -projections P and points ρ in a given spectral convex set.

Lemma 5.1. Let (A,e) and (V,K) be in spectral duality with $A = V^*$, let P be a P -projection on A and ρ an element of K such that $P^* \rho \neq 0$ and $P^* \rho \neq \rho$. Moreover, let $\sigma = \|P^* \rho\|^{-1} P^* \rho$, $\tau = \|P'^* \rho\|^{-1} P'^* \rho$ and let $\psi_F = (P + P')^*|_K$ be the unique affine retraction of K onto $\text{co}(F \cup F^\#)$ [2 ; Th.3.8]. If ρ is not contained in the line segment $[\sigma, \tau]$, then as t goes from $-\infty$ to $+\infty$ the point

$$(5.1) \quad \rho_t = \frac{\exp t(P-P')^* \rho}{\langle e, \exp t(P-P')^* \rho \rangle} \in V$$

will describe one half of the (unique) ellipse $E_P(\rho)$ through ρ which has $[\sigma, \tau]$ as one diameter and has the conjugate diameter

in the direction of the vector $\rho - \psi_F(\rho)$. If $\rho \in [\sigma, \tau]$, then as t goes from $-\infty$ to $+\infty$ the point ρ_t will describe the "degenerate ellipse" $E_P(\rho) = [\sigma, \tau]$.

Proof. We begin by giving an alternate expression for $\exp t(P-P')$. Since the two idempotents P and P' commute, an expansion of the exponential function gives

$$(5.2) \quad \exp t(P-P') = I - P - P' + \lambda P + \lambda^{-1} P'$$

where $\lambda = \exp t$ for arbitrary $t \in \mathbb{R}$.

Let $\alpha = \|P^* \rho\| = \langle e, P^* \rho \rangle$. Then we also have $1 - \alpha = \|P'^* \rho\| = \langle e, P'^* \rho \rangle$. Using (5.2), we can now express the denominator of (5.1) by the formula

$$(5.3) \quad D = \langle \lambda P e + \lambda^{-1} P' e, \rho \rangle = \lambda \alpha + \lambda^{-1} (1 - \alpha) ,$$

and we can express ρ_t by the formula

$$(5.4) \quad \rho_t = D^{-1} [(\rho - \psi_F(\rho)) + \lambda \alpha \sigma + \lambda^{-1} (1 - \alpha) \tau] .$$

Assuming $\rho \notin [\sigma, \tau]$, we introduce an affine coordinate system in the plane $[\rho, \sigma, \tau]$ such that the origin is $\pi = \frac{1}{2}(\sigma + \tau)$ and the basis vectors are

$$\epsilon_1 = \tau - \pi = \frac{1}{2}(\tau - \sigma) , \quad \epsilon_2 = \frac{1}{2}[\alpha(1 - \alpha)]^{-\frac{1}{2}}(\rho - \psi_F(\rho)) .$$

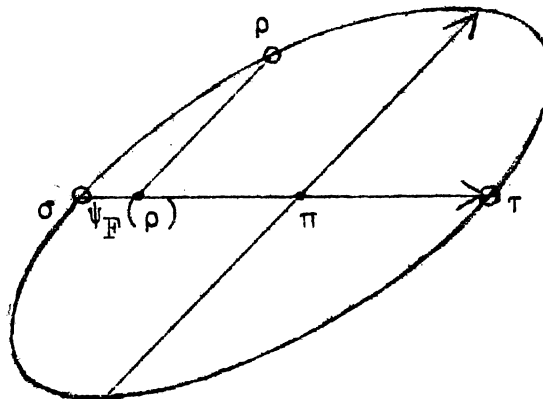


Fig. 2.

Then $\rho - \psi_F(\rho) = 2[\alpha(1-\alpha)]^{\frac{1}{2}} \epsilon_2$, $\tau = \pi + \epsilon_1$ and $\sigma = \pi - \epsilon_1$.

Substitution into (5.4) gives

$$\rho_t = \pi + \xi \epsilon_1 + \eta \epsilon_2$$

where

$$\xi = D^{-1}[\lambda^{-1}(1-\alpha) - \lambda\alpha], \quad \eta = 2D^{-1}[\alpha(1-\alpha)]^{\frac{1}{2}}.$$

Using (5.3) we find the alternate expression $\xi = 1 - 2\lambda\alpha D^{-1}$, and then also the equality $1 - \xi^2 = 4\alpha(1-\alpha)D^{-2}$. Hence the expression for ρ_t takes the form

$$(5.5) \quad \rho_t = \pi + \xi \epsilon_1 + (1 - \xi^2)^{\frac{1}{2}} \epsilon_2.$$

Now as t goes from $-\infty$ through 0 to $+\infty$, $\lambda = \exp t$ goes from 0 through 1 to $+\infty$, and we see from the expression for ξ that it will go from $+1$ through 0 to -1 . This shows that ρ_t traces out the orbit described in the statement of the Lemma.

If $\rho \in [\sigma, \tau]$, then $\psi_F(\rho) = \rho$; and now we simply obtain $\rho_t = \pi + \xi \epsilon_1$ (where ϵ_1 is defined as above). This proves the statement of the Lemma also in this case. \square

Lemma 5.2. With the assumptions and notation of Lemma 5.1,
the ellipses $E_P(\rho)$ are contained in K for all $P \in \mathcal{P}$ and
 $\rho \in K$ iff K is symmetric and $\exp t(P-P') \geq 0$ for all $P \in \mathcal{P}$
and all $t \in \mathbb{R}$.

Proof. Assume first that $E_P(\rho) \subseteq K$ for all $P \in \mathcal{P}$ and $\rho \in K$. Let $P \in \mathcal{P}$ be fixed. By Lemma 5.1, $\exp t(P-P')^* \rho \in V^+$ for every $\rho \in K$ and $t \in \mathbb{R}$. Hence $\exp t(P-P')^*$ leaves V^+ invariant, and so $\exp t(P-P')$ will leave A^+ invariant for all t . Therefore $\exp t(P-P') \geq 0$ for all $P \in \mathcal{P}$ and $t \in \mathbb{R}$.

For any given $\rho \in K$ we consider the "reflected" point

$$(2P + 2P' - I)^* \rho = 2\psi_P(\rho) - \rho \in E_P(\rho) \subseteq K.$$

Hence $(2P + 2P' - I)^*$ leaves V^+ invariant, and so $2P + 2P' - I$ leaves A^+ invariant. By definition (3.13), K is symmetric.

Assume next that K is symmetric and that $\exp t(P - P') \geq 0$ for all $P \in \mathcal{P}$ and $t \in \mathbb{R}$. Let $P \in \mathcal{P}$ and $\rho \in K$. Then $\rho_t \in K = V^+ \cap \{\omega \in V \mid \langle e, \omega \rangle = 1\}$, so one half of the ellipse $E_P(\rho)$ is contained in K . By symmetry of K the other half is also in K . \square

As before we will regard any given spectral convex set K as embedded into (V, K) in spectral duality with (A, e) where $A = V^*$, and we will use terminology from spectral theory with reference to this duality.

Definition. A spectral convex set K is elliptic if the ellipses $E_P(\rho)$ are contained in K for all $P \in \mathcal{P}$ and $\rho \in K$, or what is equivalent (by Lemma 5.2), if K is symmetric and:

$$(5.6) \quad \exp t(P - P') \geq 0 \quad \text{all } P \in \mathcal{P}, t \in \mathbb{R}.$$

Remark. One can also characterize elliptic spectral convex sets in terms of the geometry of the fibers $\psi_F^{-1}([\sigma, \tau])$ where $F \in \mathcal{F}$ and $\sigma \in F$, $\tau \in F^\#$. Such a fiber is said to have elliptic cross sections if every plane M (i.e. two-dimensional affine subspace) through $[\sigma, \tau]$ meets the fiber in an elliptic disk (i.e. the convex hull of an ellipse). Now it can be proved that a spectral convex set is elliptic iff the fibers have elliptic cross-sections. The proof is not difficult; but since we will not need this result in the sequel, we will only sketch the idea:

Assuming K elliptic, we consider a plane M through $[\sigma, \tau]$ where $\sigma \in F$, $\tau \in F^\#$ with $F \in \mathcal{F}$. Then the compact plane set $M \cap \psi_F^{-1}([\sigma, \tau])$ is a union of elliptic disks all with one diameter, and the direction of the conjugate diameter, in common. Hence $M \cap \psi_F^{-1}([\sigma, \tau])$ is itself an elliptic disk.

Assuming that K has elliptic cross sections, we consider $F \in \mathcal{F}$ corresponding to $P \in \mathcal{P}$. Let $\rho \in K$ be arbitrary, and let σ, τ be defined as in Lemma 5.1. We will prove $E_P(\rho) \subseteq E$ where E is the intersection of $\psi_F^{-1}([\sigma, \tau])$ with the plane $[\rho, \sigma, \tau]$. Being an elliptic disk, E has a unique affine retraction onto $[\sigma, \tau]$ (the projection in the direction of the conjugate diameter). But ψ_F is seen to be an affine retraction of E onto $[\sigma, \tau]$. Therefore the vector $\psi_F(\rho) - \rho$ points in the direction of the diameter of E which is conjugate to $[\sigma, \tau]$. Hence, in the two elliptic disks E and $\text{co}(E_P(\rho))$ the diameters conjugate to $[\sigma, \tau]$ are colinear. Since ρ is on the periphery of $\text{co}(E_P(\rho))$ and $\rho \in K$, we conclude that $E_P(\rho) \subseteq E \subseteq K$.

Proposition 5.3. The normal state space K of a JBW-algebra A is elliptic.

Proof. By Theorem 3.13 K is symmetric so we only have to verify (5.6) for any given P -projection P on A . Let $u = Pe$, and recall from Proposition 2.1 that $Pa = \{uau\}$ for all $a \in A$. By Macdonalds' Theorem [10; p.41], or directly from the definition of the Jordan triple product, we find for $a \in A$ and $\lambda \in \mathbb{R}^+$:

$$[\lambda P + (I - P - P') + \lambda^{-1} P']a = \{(\lambda^{\frac{1}{2}}u + \lambda^{-\frac{1}{2}}u')a(\lambda^{\frac{1}{2}}u + \lambda^{-\frac{1}{2}}u')\}.$$

By [3; Prop.2.7] the maps $a \mapsto \{bab\}$ are positive for all $b \in A$.

Therefore

$$\lambda P + (I - P - P') + \lambda^{-1} P' \geq 0 \quad \text{for all } \lambda \in \mathbb{R}^+ .$$

By formula (5.2), this completes the proof. \square

Passage to the enveloping JBW-algebra gives:

Corollary 5.4. The state space of a JB-algebra is elliptic.

§ 6. The type-I factor case.

In this section K will be a spectral convex set which we assume is imbedded as the base for a base-norm space (V, K) in spectral duality with (A, e) where $A = V^* \cong A^b(K)$; our purpose is to show that if A is a factor of type I (cf. § 1) with the pure state properties (cf. § 4), then A is a JB-algebra for the product defined by the functional calculus: $a \cdot b = \frac{1}{2}((a+b)^2 - a^2 - b^2)$. To prove this, we will eventually show that A has the key "interchange property" (2.18), and thus by Theorem 2.6 that A is a JB-algebra for this product.

We will begin by showing that for an arbitrary pair of atoms $u, v \in A$ with $u \vee v = P e$ ($P \in \mathcal{P}$), the "relativization" $A_{u,v} = \text{im } P$ (cf. Prop. 1.4) satisfies (2.18), and so is a JB-algebra. Then it follows from Proposition 5.3 that the normal state space of $A_{u,v}$ is elliptic, and we will use this to show that K is elliptic. Finally, we will use ellipticity of K together with the pure state properties to show that (2.18) holds for A , and thus that A is a JB-algebra.

Lemma 6.1. Let K be a spectral convex set with the pure state properties, and assume $A = A^b(K)$ is a factor of type I. If u, v are atoms in A and $P_0 \in \mathcal{P}$ corresponds to $u \vee v$, i.e. $u \vee v = P_0 e$, then $A_{u,v} = \text{im } P_0$ is a JBW-algebra for the product defined by the relativized functional calculus (i.e. the functional calculus in the spectral duality of $(A_{u,v}, u \vee v)$ and $(\text{im } P_0^*, K \cap \text{im } P_0^*)$, cf. Prop. 1.4).

Proof. For brevity we write $A_0 = A_{u,v} = \text{im } P_0$ and $e_0 = u \vee v = P_0 e$. We are going to show that for $P, Q \preceq P_0$:

$$(6.1) \quad [P, Q]e_0 = [Q', P']e_0.$$

By Proposition 1.4 this will show that the condition (2.18) is satisfied for every pair of P -projections on A_0 , and so by Theorem 2.6 we will have proved that A_0 is a JBW-algebra for the product mentioned above.

Write $A = Pe_0$, $b = Qe_0$, and note that a and b are both under $e_0 = u \vee v = P_0e$ since $P, Q \preceq P_0$. By Corollary 4.3, the elements $a, b, e_0 - a, e_0 - b$ will either be atoms or else equal 0 or e_0 . In the latter case (6.1) is trivially satisfied (both sides equal zero), so we will assume $a, b, e_0 - a, e_0 - b$ are all atoms.

By Proposition 1.10 the P -projections P, Q, P', Q' restricted to A_0 have one-dimensional ranges. By (1.16) we get for $c \in A_0$:

$$\begin{aligned} Pc &= \langle c, \hat{a} \rangle a, & P'c &= \langle c, (e_0 - a)^\wedge \rangle (e_0 - a), \\ Qc &= \langle c, \hat{b} \rangle b, & Q'c &= \langle c, (e_0 - b)^\wedge \rangle (e_0 - b). \end{aligned}$$

Using the equality (4.5) of the pure state properties, we find

$$[P, Q]e_0 = Pb - Qa = \langle b, \hat{a} \rangle a - \langle a, \hat{b} \rangle b = \langle b, \hat{a} \rangle (a - b)$$

and

$$\begin{aligned} [Q', P']e_0 &= Q'(e_0 - a) - P'(e_0 - b) \\ &= \langle e_0 - a, (e_0 - b)^\wedge \rangle (e_0 - b) - \langle e_0 - b, (e_0 - a)^\wedge \rangle (e_0 - a) \\ &= \langle e_0 - a, (e_0 - b)^\wedge \rangle (a - b) \\ &= (1 - \langle a, (e_0 - b)^\wedge \rangle)(a - b) \\ &= (1 - \langle e_0 - b, \hat{a} \rangle)(a - b) \\ &= \langle b, \hat{a} \rangle (a - b). \end{aligned}$$

This proves (6.1) and completes the proof. \square

In order to use Lemma 6.1 to prove results for A , we must establish that A has sufficiently many atoms. In this connec-

tion we will need the following general implication:

(6.2) If $Pa \in \text{im}^+ Q$ where $A \in A^+$ and $P, Q \in \mathcal{P}$,
then $P(r(a)) \in \text{im}^+ Q$.

To verify this, we observe that if $Pa \in \text{im}^+ Q$, then $P^{-1}(\text{im}^+ Q)$ is a weak*-closed face of A^+ containing a , and so must contain $r(a)$ by [2 ; Lem.12.2].

Lemma 6.2. Let K be a spectral convex set with the pure state properties, and assume that $A \cong A^b(K)$ is a factor of type I . Then every projective unit in A is the l.u.b. of an orthogonal family of atoms.

Proof. 1. We will first verify that e is equal to the l.u.b. $u_0 \in \mathcal{U}$ of all atoms in \mathcal{U} . (Note that \mathcal{U} is a complete lattice by [2 ; Cor.12.5], so this l.u.b. exists).

We consider an arbitrary $P \in \mathcal{P}$, and we will show that P is compatible with u_0 . Let $[u_0]$ be the order-ideal of A generated by u_0 , so $[u_0] = \text{im } Q$ where $u_0 = Qe$ with $Q \in \mathcal{P}$ [2 ; Cor.2.12]. If u_1, \dots, u_n is any finite set of atoms, then by Proposition 4.2 (ii) Pu_1, \dots, Pu_n are multiples of atoms; therefore

$$P(u_1 + \dots + u_n) \in [u_0] .$$

By the definition of the map $a \mapsto r(a)$ (see § 1), we have $r(a) \vee r(b) = r(a+b)$ for $a, b \in A^+$. Hence by the implication (6.2) above:

$$P(u_1 \vee \dots \vee u_n) = P(r(u_1 + \dots + u_n)) \in [u_0] .$$

By Proposition 2.11 of [2], this shows that $P(u_1 \vee \dots \vee u_n) \leq u_0$. Clearly the family of all l.u.b.'s of finite sets of atoms is

directed upwards with l.u.b. equal to u_0 ; by [2 ; Lem.12.1] u_0 is also the weak limit of this directed family. We have just seen that P maps each element of this family under u_0 ; by weak continuity of P we conclude $Pu_0 \leq u_0$. By Proposition 5.1 of [2] this shows that P is compatible with u_0 .

Now we have proved that u_0 is compatible with any $P \in \mathcal{P}$, and so is by definition central. Since A is a factor, either $u_0 = 0$ or $u_0 = e$. Since A is of type I , it contains at least one atom, so the first possibility is ruled out. Therefore $u_0 = e$, as desired.

2. Next we will show that for an arbitrary non-zero element $u \in \mathcal{U}$ where $u = Pe$ with $P \in \mathcal{P}$, there exists at least one atom w such that $w \leq u$. If v is any atom, then by Proposition 4.2 Pv is either a non-zero multiple of an atom or $Pv = 0$. In the former case $w = \|Pv\|^{-1} Pv = r(Pv)$ is an atom satisfying $Pv \leq Pe = u$, so our claim is verified. We will show that the latter case can not prevail for all atoms $v \in \mathcal{U}$. In fact, if $Pv = 0$ where $v = Qe$ with $Q \in \mathcal{P}$, then $PQe = 0$, which implies $P \perp Q$, and so $u \perp v$ (cf. [2 , p.28]). Thus, if $Pv = 0$ for all atoms v , then $v \leq u' = e - u$ for all atoms v ; since e has been shown to be the l.u.b. of atoms, this gives $e \leq e - u$, i.e. $u = 0$, contrary to assumption.

3. Again we consider an arbitrary non-zero element $u \in \mathcal{U}$. By Zorn's lemma there exists a maximal orthogonal family $\{u_\alpha\}$ of atoms under u . Let $v \in \mathcal{U}$ be the l.u.b. of $\{u_\alpha\}$. Now we must have $u = v$, for otherwise $u - v$ would contain an atom which would be orthogonal to all u_α , contrary to the maximality of $\{u_\alpha\}$. This completes the proof. \square

Lemma 6.3. Let K be a spectral convex set with the pure state properties, and assume that $A \cong A^b(K)$ is a factor of type I . Then K is elliptic.

Proof. By the definition of ellipticity (§ 5) we have to prove that for any fixed $P \in \mathcal{P}$:

$$(6.3) \quad 2P + 2P' - I \geq 0 ,$$

$$(6.4) \quad \exp t(P - P') \geq 0 \quad \text{for all } t \in \mathbb{R} .$$

We will first show that each of these maps sends an arbitrary atom $u \in \mathcal{U}$ into A^+ . Note first that if $Pu = 0$ then $u \in \ker^+ P = \text{im}^+ P'$, so $P'u = u$, and one easily sees that $(2P + 2P' - I)u \geq 0$ and $\exp t(P - P')u \geq 0$ for all $t \in \mathbb{R}$. Thus we assume $Pu \neq 0$. By Proposition 4.2 Pu is a non-zero multiple of an atom $v \in \mathcal{U}$. Let $Q \in \mathcal{P}$ correspond to uvv , i.e. $uvv = Qe$, and let $A_{u,v} = \text{im} Q$. We claim that P and P' leave $A_{u,v}$ invariant. In fact, since $P(u+v)$ is a multiple of v , this element will belong to the order ideal $[uvv] = \text{im} Q$ (cf. [2 ; Cor.2.12]; therefore we can apply (6.2) to get

$$P(uv v) = P(r(u+v)) \in \text{im}^+ Q .$$

Now it follows from Proposition 2.11 of [2] that $P(uv v) \leq uv v$. Hence P , and then also P' , will be compatible with $uv v$ (cf. [2 ; Prop.5.1]) . Therefore P and P' will commute with Q [2 ; Prop.5.2], and so they leave $A_{u,v} = \text{im} Q$ invariant, as claimed.

Now it follows by Proposition 1.4 that P and P' restricted to $A_{u,v}$ are quasicomplementary P -projections (with respect to the "relativized spectral duality"). Since $A_{u,v}$ is known to

be a JBW-algebra, its normal state space (i.e. $K \cap \text{im } Q^*$) is elliptic. From this it follows that $(2P + 2P' - I)u \in A_{u,v}^+ \subseteq A^+$, and that $\exp t(P - P')u \in A_{u,v}^+ \subseteq A^+$ for all $t \in \mathbb{R}$.

Clearly the maps in (6.3) and (6.4) are positive also on orthogonal sums of atoms, and by weak* continuity and by Lemma 12.1 of [2] they are positive on l.u.b.'s of such sums. By Lemma 6.2 every $u \in \mathcal{U}$ is a l.u.b. of this type, so the maps in question are positive on all projective units. Finally, it follows by spectral theory and norm continuity, that these maps are positive on all elements $a \in A^+$, which completes the proof. \square

Notation: Assuming, as before, that K is a spectral convex set, we will use the symbol A_f to denote the linear span of all atoms in $A \cong A^b(K)$.

Lemma 6.4. Let K be a spectral convex set with the pure state properties. Then there exists a unique symmetric bilinear form (\mid) on A_f such that

$$(6.5) \quad (u \mid v) = \langle u, \hat{v} \rangle$$

for all pairs of atoms u, v . This form is norm continuous in each variable separately. Furthermore, every $P \in \mathcal{P}$ will map A_f into itself, and the restriction of P to A_f is symmetric with respect to this form, in that

$$(6.6) \quad (Pa \mid b) = (a \mid Pb)$$

for all pairs $a, b \in A_f$.

Proof. 1. Consider first an atom $u \in \mathcal{U}$ and an element $b \in A_f$, say $b = \sum_{j=1}^m v_j v_j$ where v_1, \dots, v_m are atoms. Then by

equation (4.5) of the pure state properties:

$$\sum_{j=1}^m v_j \langle u, \hat{v}_j \rangle = \sum_{j=1}^m v_j \langle v_j, \hat{u} \rangle = \langle b, \hat{u} \rangle .$$

Hence the expression $\sum_{j=1}^m v_j \langle u, \hat{v}_j \rangle$ depends only on u and b and not on the way in which b is represented. By linearity, this result will subsist if we replace u by an arbitrary element of A_f , say $a = \sum_{i=1}^n \lambda_i u_i$ where u_1, \dots, u_n are atoms. Thus, there is a well defined bilinear form on A_f given by

$$(6.7) \quad (a|b) = \sum_{j=1}^m \sum_{i=1}^n \lambda_i v_j \langle u_i, \hat{v}_j \rangle = \sum_{j=1}^m v_j \langle a, \hat{v}_j \rangle .$$

Clearly, this form satisfies the requirement (6.5), and symmetry follows by application of (4.5) to the terms $\langle u_i, \hat{v}_j \rangle$ of (6.7).

2. For fixed $b \in A_f$ the map $a \mapsto (a|b)$ is norm continuous since each \hat{v}_j occurring at the right side of (6.7) is a continuous linear functional. Hence the bilinear form $(|)$ is norm continuous in the second variable. By symmetry, it is also norm continuous in the first variable.

3. Let $P \in \mathcal{P}$ be fixed. By Proposition 4.2 P preserves atoms, hence it leaves A_f invariant. In order to prove that P is symmetric with respect to the bilinear form on A_f , we start by establishing

$$(6.8) \quad ((I-P)u | Pv) = 0$$

for an arbitrary pair of atoms u, v .

By Proposition 4.2 P is a multiple of an atom, say $Pv = \lambda w$ where $\lambda \in \mathbb{R}^+$ and $w = Qe$ for a P -projection Q with one-dimensional range (cf. Prop.1.10). Now, $\text{im}^+ Q$ is the ray

determined by w , and since $w \in \text{im}^+ P$ we must have $\text{im}^+ Q \subseteq \text{im}^+ P$, which shows that the projective face corresponding to Q is contained in the projective face corresponding to P [2; Lem.2.16]. By definition, \hat{w} is the unique element of the projective face $K \cap \text{im} Q^*$ corresponding to Q . Hence $\hat{w} \in K \cap \text{im} P^*$, and so $P^* \hat{w} = \hat{w}$.

Using this and the definition of the bilinear form on A_f , we obtain

$$\begin{aligned} ((I-P)u | Pv) &= \lambda((I-P)u | w) = \\ &= \lambda \langle (I-P)u, \hat{w} \rangle = \lambda \langle u, (I-P)^* \hat{w} \rangle = 0, \end{aligned}$$

which establishes (6.8).

From (6.8) we conclude that

$$(Pu | v) = (Pu | Pv) = (u | Pv)$$

for every pair of atoms u, v . Now (6.6) follows by linearity, since by definition A_f is the linear span of atoms. \square

We are now ready for the main result of this section.

Theorem 6.5. Let K be a spectral convex set with the pure state properties and assume that $A \cong A^b(K)$ is a factor of type I. Then A is a JB-algebra for the product $a \circ b = \frac{1}{2}((a^2 + b^2) - a^2 - b^2)$.

Proof. Let $P, Q \in \mathcal{P}$ be arbitrary. By Theorem 2.6 and Lemma 2.3 we shall be through if we can prove

$$[P-P', Q-Q']e = 0,$$

which is equivalent to

$$(6.9) \quad (\exp t[P-P', Q-Q'])e = e \quad \text{for all } t \in \mathbb{R}.$$

To prove (6.9) we will show that the maps $U_t = \exp t[P-P', Q-Q']$ with $t \in \mathbb{R}$ are affine automorphisms of the cone A_f^+ , then show that they preserve the bilinear form introduced in Lemma 6.4, and finally use these properties of the maps U_t to show that they leave the order-unit e invariant.

We first recall the following known identity valid for arbitrary operators R, S on a Banach space:

$$(6.10) \quad \lim_{n \rightarrow \infty} [\exp(-\frac{t}{n}R) \exp(-\frac{t}{n}S) \exp(\frac{t}{n}R) \exp(\frac{t}{n}S)]^{n^2} = \\ = \exp t^2 [R, S] \quad \text{for all } t \in \mathbb{R}.$$

(Cf. e.g. [9 ; p.96 and p.105] where the relevant series expansions are given in a finite dimensional context which admits a straightforward generalization to bounded operators on a Banach space).

By Lemma 6.3 K is elliptic, so the maps $\exp t(P-P')$, $\exp t(Q-Q')$ send A^+ into itself for all $t \in \mathbb{R}$. Then it follows by (6.10) that the maps U_t send A^+ into itself for all $t \geq 0$. Repeating the same argument with P and P' interchanged, we conclude that the maps U_t send A^+ into itself also for $t < 0$. Since U_t is invertible with inverse U_{-t} , it follows that U_t must be an affine automorphism of A^+ for every $t \in \mathbb{R}$. From this it follows, in particular, that every U_t will map the extreme rays of the cone A^+ onto extreme rays. By the spectral theorem [2 ; Th.6.8] and Proposition 1.10, it follows that the extreme rays of A^+ are precisely the rays R^+u where u is an atom. Thus, every U_t will map atoms to multiples of atoms, and so leave A_f invariant. This shows that U_t acts as an affine automorphism of the cone A_f^+ for every $t \in \mathbb{R}$.

From now on we fix $t \in \mathbb{R}$. Note that by Lemma 6.4 the op-

rators $P-P'$ and $Q-Q'$ are symmetric with respect to the bilinear form $(|)$ on A_f , therefore their commutant is "skew". More specifically, for every pair $a, b \in A_f$ we obtain by (6.6)

$$(6.11) \quad ([P-P', Q-Q']a|b) = -(a|[P-P', Q-Q']b) .$$

Writing $T = [P-P', Q-Q']$, we now obtain

$$(U_t a|b) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (T^n a|b) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} (a|T^n b) = (a|U_{-t} b) ,$$

and then also

$$(U_t a|U_t b) = (a|U_{-t} U_t b) = (a|b) .$$

This shows that U_t preserves the bilinear form $(|)$ on A_f .

We have previously shown that U_t maps atoms to multiples of atoms; we now show that atoms are actually mapped to atoms. Fix an atom $u \in \mathcal{U}$; then for some $\lambda \in \mathbb{R}^+$ we have $U_t u = \lambda v$ where $v \in \mathcal{U}$ is an atom. Now

$$1 = (u|u) = (U_t u|U_t u) = \lambda^2 (v|v) = \lambda^2 ,$$

so $\lambda = 1$, showing that $U_t u$ is equal to the atom v .

We now claim that for atoms, spectral orthogonality is equivalent to orthogonality defined in terms of the bilinear form $(|)$. To verify this, we consider atoms $u, v \in \mathcal{U}$. If u is orthogonal to v in the spectral sense, then (by [2; (4.6)]) $u+v \leq e$, so $0 \leq (u|v) = \langle u, \hat{v} \rangle \leq \langle e-v, \hat{v} \rangle = 0$. Conversely, if $(u|v) = 0$, then $\langle e-u, \hat{v} \rangle = 1$, so by [2; (2.19)] $v \leq e-u$, i.e. u and v are orthogonal in the spectral sense.

From what we have just proved, it follows that U_t maps orthogonal atoms to orthogonal atoms.

We are now ready to show that $U_t e = e$. By Lemma 6.2 there exists an orthogonal family of atoms $\{u_\alpha\}$ with l.u.b.

equal to e . For each index α we define $v_\alpha = U_t u_\alpha$. Then $\{v_\alpha\}$ is an orthogonal family of atoms. For every finite set J of indices we have (cf. (1.9)):

$$U_t \left(\sum_{\alpha \in J} u_\alpha \right) = \sum_{\alpha \in J} v_\alpha \leq e.$$

Now $\sum_{\alpha \in J} u_\alpha$ converges weakly to e as J increases [2; Lem.12.1]; by weak continuity of U_t we get

$$U_t e = \lim_J U_t \left(\sum_{\alpha \in J} u_\alpha \right) = \lim_J \sum_{\alpha \in J} v_\alpha \leq e.$$

This holds for $-t$ as well as t , so $U_{-t} e \leq e$, which by positivity of U_t gives $e = U_t(U_{-t} e) \leq U_t e$. Hence $U_t e = e$. Since $t \in \mathbb{R}$ was arbitrary, this establishes (6.9), and the proof is complete. \square

The following corollary will not be needed in the sequel, but may be of some interest in itself.

Corollary 6.6. A convex set K is affinely isomorphic to the normal state space of a JB-factor of type I iff it satisfies conditions (i) and (ii) and either one of (iii) and (iii)' below:

- (i) K contains no proper split faces and has at least one extreme point
- (ii) K is spectral
- (iii) K is symmetric and has the Hilbert ball property.
- (iii)' K has the pure state properties.

Proof. The conditions are necessary by Proposition 2.1, Theorems 3.11 and 3.14, Propositions 4.1 and 4.4, and by the

definition of a factor of type I. (Recall the connections between central projections and split faces [2 ; Prop.10.2], and between atoms and exposed points, cf. Proposition 1.10).

The conditions (i), (ii), (iii)' together will be sufficient by Theorem 6.5; condition (iii) implies (iii)' (again by Propositions 4.1 and 4.4), which completes the proof. \square

§ 7. Characterization of state spaces of JB-algebras.

In this section we will combine our previous results and establish the main theorem characterizing state spaces of JB-algebras. As before, every spectral convex set K is assumed to be imbedded as the base for a base-norm space (V, K) in spectral duality with (A, e) where $A = V^* \cong A^b(K)$. When K is a compact convex set, we will use the notation $A(K)$ for the space of all continuous affine functions on K . The crucial point in the development will be to prove that under appropriate hypotheses on a strongly spectral compact convex set K , the space $A(K)$ is a JB-algebra for the product defined by the functional calculus, i.e.

$$(7.1) \quad a \cdot b = \frac{1}{2}((a+b)^2 - a^2 - b^2) .$$

We will show that the product (7.1) is bilinear, and then that $A(K)$ is a JB-algebra. Our procedure will be to use the fact that $A(K)$ has a separating set of "type-I factor representations". By the results of § 6, the product (7.1) will be bilinear modulo each such representation, and therefore will be bilinear in general.

The following Lemma will be needed.

Lemma 7.1. Let (A, e) and (V, K) be in spectral duality. If $P \in \mathcal{P}$ and $a \in A$ are compatible, then $P(a^2) = (Pa)^2$.

Proof. Since P is compatible with a , it is compatible with all its spectral units e_λ ; therefore $Pe_\lambda = (Pe) \wedge e_\lambda \in \mathcal{U}$ for all $\lambda \in \mathbb{R}$ (cf. [2; Prop.5.2]). Defining $f_\lambda = Pe_\lambda$ for $\lambda < 0$ and $f_\lambda = Pe_\lambda + P'e$ for $\lambda \geq 0$, we get a spectral family; this family must be the (unique) spectral resolution for Pa

since

$$Pa = \int \lambda dPe_\lambda = \int \lambda df_\lambda .$$

By the definition of squares

$$(Pa)^2 = \int \lambda^2 df_\lambda = \int \lambda^2 dPe_\lambda = P(\int \lambda^2 de_\lambda) = P(a^2) ,$$

which proves the lemma. \square

Theorem 7.2. If K is a strongly spectral compact convex set then the following are equivalent:

- (i) K is symmetric and has the Hilbert ball property
- (ii) K has the pure state properties
- (iii) $A(K)$ is a JB-algebra for the product (7.1)
- (iv) $A = A^b(K)$ is a JBW-algebra for the product (7.1).

Proof. (i) \Rightarrow (ii) was proven in § 4.

To prove (ii) \Rightarrow (iii) we assume the pure state properties. For a fixed extreme point $\rho \in K$ we consider the (central) P -projection P corresponding to $c(\rho)$, and we recall from Proposition 1.9 that $A_\rho = \text{im } P$ is a factor with respect to the relativized spectral duality of $(A_\rho, c(\rho))$ and (V_ρ, F_ρ) where $V_\rho = \text{im } P^*$ and $F_\rho = K \cap \text{im } P^*$. Note that the atoms of A_ρ are exactly those atoms of A which are contained in A_ρ (since A_ρ^+ is a face of A^+), and that the extreme points of F_ρ are exactly those extreme points of K which are contained in F_ρ (since F_ρ is a face of K). Therefore the pure state properties in the relativized duality will follow from those in the given duality (cf. also Proposition 1.4). Furthermore, A_ρ contains at least one atom, namely the atom corresponding to the minimal projective

face $\{\rho\}$ of F_ρ (cf. (4.3) of the pure state properties).

Hence A_ρ is a factor of type I. Now it follows from Theorem 6.5 that A_ρ is a JB-algebra for the product (7.1).

By the definition of A_ρ the elements of this space are of the form Pa for $a \in A$, and it follows from Lemma 7.1 that the product determined by the functional calculus on A_ρ is given by

$$(7.2) \quad (Pa) \circ (Pb) = P(a \circ b) \quad \text{for } a, b \in A.$$

This product in the JB-algebra A_ρ is bilinear; therefore we have the following identity for $a, b, c \in A$:

$$\begin{aligned} P(a \circ (b+c) - a \circ b - a \circ c) &= \\ &= (Pa) \circ (Pb + Pc) - (Pa) \circ (Pb) - (Pa) \circ (Pc) = 0. \end{aligned}$$

Applying the state $\rho = P^* \rho$ to the element $a \circ (b+c) - a \circ b - a \circ c$, we now conclude

$$(7.3) \quad \langle a \circ (b+c) - a \circ b - a \circ c, \rho \rangle = 0.$$

From now on we assume that a, b, c are in $A(K)$ and not merely in $A = A^b(K)$. Since K is strongly spectral, the space $A(K)$ is closed under squaring, and hence under the product (7.1). Therefore the element at the left side of (7.3) will also be in $A(K)$. Since the identity (7.3) holds for an arbitrary extreme point ρ of K , it now follows from the Krein-Milman theorem that it holds for every $\rho \in K$. Thus, $a \circ (b+c) = a \circ b + a \circ c$. Similarly we prove $a \circ (\lambda b) = \lambda a \circ b$, and the Jordan identity $(a^2 \circ b) \circ a = a^2 \circ (b \circ a)$. This shows that (7.1) defines a product making $A(K)$ a Jordan algebra, and so by Theorem 2.1 of [3], $A(K)$ becomes a JB-algebra for this product.

To prove (iii) \Rightarrow (iv) we assume that $A(K)$ is a JB-alge-

bra for the product (7.1). By [20; Ths.1.2 and 1.4] the bidual $A(K)^{**} \cong A^b(K)$ will be a JBW-algebra for the Arens product, and can be identified with the enveloping algebra $A(K)^\sim$. Now by [2; Th.12.13] it follows that the Arens product must coincide with the product (7.1), which proves (iv).

Finally, the implication (iv) \Rightarrow (i) follows by Theorems 3.11 and 3.14 since K can be identified with the normal state space of the JBW-algebra $A^b(K)$ (cf. [20; Th. 2.3]). \square

The main theorem is now an easy consequence of preceding results.

Theorem 7.3. A compact convex set K is affinely and topologically isomorphic to the state space of a JB-algebra (with the weak*-topology) iff K is symmetric and strongly spectral and has the Hilbert ball property.

Proof. The conditions are sufficient by the implication (i) \Rightarrow (iii) of Theorem 7.2, and they are necessary by Corollaries 2.2, 3.12 and 3.15. \square

References

1. E.M. Alfsen, Compact convex sets and boundary integrals. *Ergebnisse der Math.* 57, Springer Verlag, Berlin 1971.
2. E.M. Alfsen and F.W. Shultz, Non-commutative spectral theory for affine function spaces on convex sets. *Memoirs Amer. Math. Soc.*, No. 172, Providence R.I., 1976.
3. E.M. Alfsen, F.W. Shultz and E. Størmer, A Gelfand-Neumark theorem for Jordan algebras. *Advances in Math.* (to appear).
4. G. Birkhoff, Lattice theory. *Amer. Math. Soc. Colloq. Publ.*, Vol. 25, 3rd. edition, Providence, R.I. 1967.
5. H. Braun und M. Koecher, *Jordan Algebren*. Springer Verlag, Berlin 1966.
6. A. Connes, Caracterisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann. *Ann. Inst. Fourier, Grenobles*, 24 (1974), 121-155.
7. E.B. Davies and J.T. Lewis, An operational approach to quantum probability. *Communications Math. Phys.* 17 (1970), 239-260.
8. J. Gunson, On the algebraic structure of quantum mechanics. *Communications Math. Phys.* 6 (1967), 262-285.
9. S. Helgason, *Differential geometry and symmetric spaces*. Academic Press, New York 1962.
10. N. Jacobson, *Structure and representations of Jordan algebras*. *Amer. Math. Soc. Colloq. Publ.* Vol. 39, Providence R.I. 1968.
11. P. Jordan, J. von Neumann and E. Wigner, On an algebraic generalization of the quantum mechanical formalism. *Annals of Math.* 35 (1934), 29-64.
12. D. Lowdenslager, On postulates for general quantum mechanics. *Proc. Amer. Math. Soc.* 8 (1957), 88-91.

13. F. Maeda and S. Maeda, Theory of symmetric lattices. Grundlehren der math. Wissensch., No. 173, Springer Verlag, Berlin 1970.
14. B. Mielnik, Theory of filters. Communications Math. Phys. 15 (1969), 1-46.
15. J. von Neumann, On an algebraic generalization of the quantum mechanical formalism, Part I. Mat. Sbornik 1 (1936), 415-484.
16. J.C.T. Pool, Semimodularity and the logic of quantum mechanics. Communications Math. Phys. 9 (1968), 218-228.
17. E. Schreiner, Modular pairs in orthomodular lattices. Pacific J. Math. 19 (1966), 519-528.
18. I.E. Segal, Postulates for general quantum mechanics. Annals of Math. 48 (1947), 930-948.
19. S. Sherman, On Segal's postulates for general quantum mechanics. Annals of Math. 64 (1956), 593-601.
20. F.W. Shultz, On normed Jordan algebras which are Banach dual spaces. Preprint.
21. D. Topping, An isomorphism invariant for spin factors. J. Math. and Mech. 15 (1966), 1055-1064.
22. D. Topping, Asymptoticity and semimodularity in projection lattices. Pacific J. Math. 20 (1967), 317-325.
23. W. Wils, The ideal center of a partially ordered vector space. Acta Math. 127 (1971), 41-79.